POSSIBLE AND IMPOSSIBLE MELODY: SOME FORMAL ASPECTS OF CONTOUR

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Preface

This paper explores some formal aspects of contour, especially the mathematics of an abstract definition of contour itself. In the hope of establishing a general formulation that will be of use to more specific, style- and genre-related theoretical work in contour, a "non-stylistic" approach is taken. Specific musical situations (like the equivalence classes generated by elementary transformations, or musical assumptions made by ethnomusicological contour studies) are not invoked.

Further generalization of the theory of "the number of possible contours" includes the formulation of a theory of contour for *asymmetrical* and *non-ternary* contour descriptions, one we believe to be of musical interest.

We assume that contour may be applied to any parameter of music, at any hierarchical level. These ideas may be used in the analysis of waveforms, melody, the sequence of pitch means in some large-scale segmentation of a piece, or any other quantifiable parameter.

I. Introduction: Definitions and Uses of Contour

The idea of contour is of considerable importance in music theory, composition, experimental psychology, ethnomusicology, and perception. Psychologists (Dowling, Edworthy, Cuddy, and others) have found the perception of contour to be integral to melodic memory and recognition. Various definitions of *contour* and *direction* are the basis for descriptions and analyses of the musics of Central Java, Native America, and other cultures (Kolinski, Becker, Adams, Hastanto, Seeger, and others). Seeger succinctly points out the importance of contour and emphasizes its generative role when he describes what he calls "musical logic" as "more the tool of the producer of music than of the student of the product."

In recent theoretical literature on western music, Morris, Friedmann, Marvin, and Laprade have exploited more formal definitions of contour in an atonal "set-theoretic" context, specifically in the analysis of the music of Schoenberg and Webern. Polansky (1990, 1988, 1987) has used formal definitions of contour and contour metrics as an important determinant of compositional form, especially in the context of computer-aided composition and performance.

Most authors have proposed their own definitions of contour, usually specific to the experiment or form of analysis. Many of these definitions are functionally equivalent, but necessarily use different notations. This is to some extent the case with the relationship of this paper to some of the work of Morris, Laprade and Marvin, and Friedmann. This natural diversity of description even prompted Friedmann (1987) to write a kind of lexicon of contour notation and similarity functions ("My Contour, Their Contour")—we hope that the present article, with its various notations, will not necessitate even further excursions into the possessive pronoun.

In the literature of experimental psychology, contour has usually been defined in a *linear* way, as sequential relationships between elements of some ordered set. It is generally presumed that the listener is paying particular attention to, for example, the "moment by moment" direction, or "pattern of ups and downs" of a melody (Dowling, 1978), and not to the network of contour relationships between the various pitches.[1] In most cases, this simple definition of (melodic) contour makes sense: listeners are often most sensitive to adjacency relations, and they tend to forget non-adjacent contour relationships quickly. In experimental psychology, the use of a linear definition of contour substantially limits the possible complexity of stimuli and permits significant research on contour's role in melodic recognition, memory, and general perception.

In the ethnomusicological (Adams, Kolinski, Seeger), and especially in the theoretical literature (Morris, Friedmann,[2] Marvin and Laprade), authors have tended to be more concerned with what Polansky has called *combinatorial contour* (1987). It is important to state that here we are not using the word *combinatorial* in reference to the rich literature on atonal set theory, in which it has a different connotation. This paper shares a concern with combinatorial contour with some of the other works mentioned, but the methodology and notation here are motivated by a specific set of theoretical questions.

II. Combinatorial Contour

A simple definition of contour, in accord with the existing literature, assigns a value of +, −, or =, or 1, −1, or 0 to the interval between two elements, depending on whether or not the first element is greater to, less than, or equal to the second, in some measurable parameter. This can be called a three-valued, or *ternary symmetrical* contour description. It is ternary because the "grain" of distinction, −1, 0, 1, only permits three possible comparisons between two elements (the relation function can assume three distinct values). This description is symmetrical because there are as many ways for something to be *less than* something else as there are for something to be *greater than* something else.

A two-valued, or *binary* (symmetrical) contour, omitting equality, is used by some authors, especially in experimental psychology. *No change* along a certain parameter will often mean that two successive elements, in a specified (usually perceptual) context, are the same. For example, Friedmann's *CAS,* which he proposes as a preliminary "rather blunt, general description of a series of moves between temporally adjacent pitches" (1985) is an example of a binary contour description. Binary contours have often been used in ethnomusicological studies, where repeated notes are considered to be one element. Since in the present article we are not assuming any specific musical context (as, for example, Friedmann was with his Schoenberg analysis), or parameter (our contour descriptions are in no way limited to pitch morphologies), we cannot necessarily eliminate equality from the set of relations.[3]

Most authors have used symmetrical contour descriptions, for obvious reasons. It is difficult to think of examples of asymmetrical binary contour descriptions—only indicating *change or no change*—but one interesting instance is a form of what is known in digital signal processing as *delta modulation,* where an incoming audio signal is encoded as a one-bit sequence of "change values." An asymmetrical situation with, for example, four relations (*a lot less than, a little less* *than, equal to,* and *greater than*) is less common than a symmetrical one (and perhaps unnecessarily overlooked). We intend that the formal descriptions presented here can be generalized to asymmetric and non-ternary contours as well.

Linear and Combinatorial Contour

Polansky (1987) has described a simple contour function that assigns a value of −1, 1, or 0 to two elements depending on their relationship, calling it *sgn(a,b)*.[4] The value 1 is assigned to the relation: "a is greater than b," −1 to "a is less than b," and 0 to "a is equal to b." This function, and others used here, are more precisely defined in the mathematical proof given below. Similar "direction" functions, e.g., Marvin and Laprade's *COM(a,b),* have been used by other authors. Assuming that it is possible to determine these three relationships between any two elements, it is possible to use these simple functions to describe an ordered sequence of values, or a *morphology,* by its contour (Polansky, 1987, 1978; Tenney, 1961).[5] The *length* (L) of a morphology is simply the number of elements in the set. For example, the following set of durations (the first measure of "The Stars and Stripes Forever")

can be described in the following way as the contour relations between adjacent elements, where the elements are labelled in order {a,b, c,d,e}:

{1, 1, −1, −1} {sgn(a,b), sgn(b,c), sgn(c,d), sgn(d,e)}

For a morphology of length L there are of course $L-1$ adjacent relationships.

This type of description is called *linear,* because it only considers adjacent relationships. A *combinatorial* description uses relations between non-adjacent elements of the morphology, usually in addition to the adjacent relations (Polansky, 1987). Generally, combinatorial descriptions contain more information than linear ones and can be used to "take metrics" (measure similarity) between morphologies with a greater degree of sensitivity (Polansky, 1987; Marvin and Laprade).[6] Similar notions of linearity and combinatoriality (though with different terminology) are used by Morris, Friedmann (1987, 1985), and Marvin and Laprade in their work on pitch contours.[7]

A *combinatorial description* of the above Sousa duration sequence could be represented by the following matrix:

0	1	1	1	0	{sgn(a,a), sgn(a,b), sgn(a,c), sgn(a,d), sgn(a,e)}
−1	0	1	1	−1	{sgn(b,a), sgn(b,b), sgn(b,c), sgn(b,d), sgn(b,e)}
−1	−1	0	−1	−1	{sgn(c,a), sgn(c,b), sgn(c,c), sgn(c,d), sgn(c,e)}
−1	−1	1	0	−1	{sgn(d,a), sgn(d,b), sgn(d,c), sgn(d,d), sgn(d,e)}
0	1	1	1	0	{sgn(e,a), sgn(e,b), sgn(e,c), sgn(e,d), sgn(e,e)}

The rows and columns of this matrix can of course be arranged in a number of equivalent ways: a likely alternative would construe the first row as {sgn(a,b), sgn(b,c) . . .}, the second row as {sgn(a,c), sgn(b,d) . . .}, etc. (see Marvin and Laprade's INT_n function for a very specific implementation of this type of matrix construction).

Description of Contour as a Ternary Number

We use here a slightly different notation for ternary symmetric contours: *ternary numbers*. Instead of using the conventional (−,=,+), or (−1,0,1) to describe contours, we will use *base three* (0,1,2), to describe (a<b, a=b, a>b) respectively. The discussion that follows will clarify this choice of notation. In using ternary (base 3) numbers to describe contour, 0 means "is less than," 1 means "is equal to," and 2 means "is greater than." The placement of the 2 and 0 is arbitrary: everything that follows would work if the symbols for "greater than" and "less than" were reversed. The matrix above, written in ternary notation, is as follows:

1	2	2	2	1
0	1	2	2	0
0	0	1	0	0
0	0	2	1	0
1	2	2	2	1

Binomial Coefficient

For a set of ordered values, or *morphology,* of length L, there are in general

$$L_m = (L^2 - L)/2$$

values necessary to describe the contour. L_m describes the "matrix length" of a morphology L. This number, in simpler terms, defines the "number of relations" between L objects. It is called the *binomial coefficient* (Knuth, p. 52) and describes how, in this instance, "L elements can be taken 2 at a time".[8]

The binomial coefficient is an obvious result of two conditions that make most of the matrix unnecessary:

1) *the diagonal is always one (things are equal to themselves)*

2) *"half the matrix minus the diagonal" is redundant in terms of contour: if a<b, then b>a.*

These conditions apply specifically to the above formulation of the matrix, the specific "positions" of values will change if the matrix is ordered in some other way.

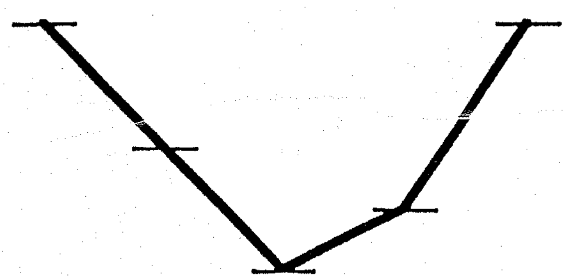
From the above, to describe a morphology of length L, there are $(L^2-L)/2$ values needed. We call the length of the morphology *L,* and the binomial coefficient, L_m. For L=3, L_m =3; L=4, L_m =6; L=5, L_m =10; L=6, L_m = 15; L =7, L_m = 21; and so on. In general, the *number of possible symmetric ternary contour descriptions* is 3 *raised to the binomial coefficient power.* This corresponds to all the possible half-matrices (minus the diagonal) of ternary contours. For example, where L=4 (L_m=6), there are

$$3^{Lm} = 3^6 = 729$$

possible contour descriptions. For L=3, there are 3^{Lm} or 3^3 possible descriptions of contours, consisting of all the ternary numbers of 3 digits (000, 001, 002, 010, etc.). That is, there are 27 ternary contour descriptions.

A simple construction for L_m-digit ternary numbers from a matrix is as follows. First list all the contour relations from the first element of the morphology to all the others (first row of the matrix starting in column two); then from the second element to all the others (first row of the matrix starting in column two); then from the second element to all others *after* it (a second row of the matrix, starting in the third column), and so on.[9] A slightly more formal description of this process is given in the proof below. This simply consists of "writing out the half-matrix linearly, row by row," starting with the first non-redundant, non-equal cell in each row. The rhythmic sequence from the Sousa example would be written, for example, as

2221 220 00 0



Inserted spaces indicate where each element of the morphology begins to compare to the ones *after* it. The first value is greater than the second, third, and fourth, and equal to the fifth; the second value is greater than the third and the fourth, but less than the fifth; the third value is less than the fourth and fifth; the fourth value is less than the fifth.

A simpler example is the following three-element pitch morphology:

The resulting three-digit ternary number is 002 ("C to F, C to E♭, F to E♭"). For a slightly longer example, where L = 6 ($L_m = 15$)

we obtain the following fifteen-digit ternary number, a full description of the combinatorial contour of the morphology:

22222 2101 000 01 2

which is a linear transcription of the matrix:

2	2	2	2	2
	2	1	0	1
		0	0	0
			0	1
				2

There is a simple way to read these ternary numbers examples: first read the first L−1 digits, then the next L−2 digits, and so on, down to the last digit, which is the contour relation between the last two elements in the morphology.

Impossible Contours

For a given L_m, there are many more *possible* combinatorial contour descriptors than there are actual contours. For example, the following ternary contour descriptor:

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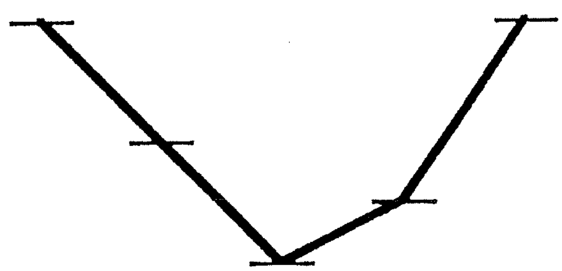
—*can not exist.* As the diagram shows, it implies that the first element descends to the second, and is equal to the third, but that the third is equal to the second, which *violates transitivity*.

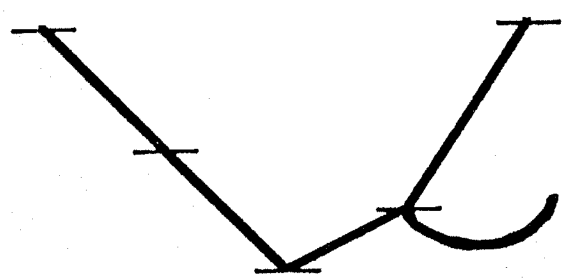
As another example of impossible contour, we can take the Sousa example contour number

2221220000



and change the last digit to make it impossible:

22**21** 220 00 **1**



The highlighted values show the values that cause an impossibility. The first element is greater than the fourth, but equal to the fifth, but the fourth equals the fifth: a = b, b = c, a > c.

In general, it is not simple to discern *possibility* from combinatorial contour descriptors with morphologies of greater than length three—at least we don't know of a simple algorithm yet! That is, it is relatively straightforward to design a computer algorithm which will list all possible contours, or test the possibility of a given ternary number, but these algorithms tend to be "brute force," testing the number in a manner similar to having to draw the contour by hand until a violation of transitivity is reached or not.[10] Certain rules, however, based on simple logic and easy to formulate, give quicker evaluations of contour possibility, and eliminate certain numbers on the basis of their numerical patterns. Where L=3, for example, there can never be exactly two 1's in the description, as shown above. This is an obvious result of the fact that if two of the three possible relationships show equality, so must the third. With ternary contours, another less obvious, necessary, but not sufficient condition is that the sum of the first and third digits must be greater than or equal to the second. The formulation of these kinds of rules is specific to the way in which the contours are described—the second would not hold, for example, if {−1,0,1} were used instead of ternary numbers. The formulation of rules of this type for L>3 remains an interesting problem.

What causes an *impossible contour* is a *violation of transitivity,* that is, a situation like:

a>b, b>c, c>a[11]

The general formulation of inclusion rules for possible contour requires that a simple and elegant set of tests for transitivity be devised for n-digit ternary numbers that describe these half-matrixes. It is easy to design algorithms that list all the possible contours,[12] but this is a different task from being able to decide, from a given contour, whether or not it is possible.

III. The Number of Possible Contours

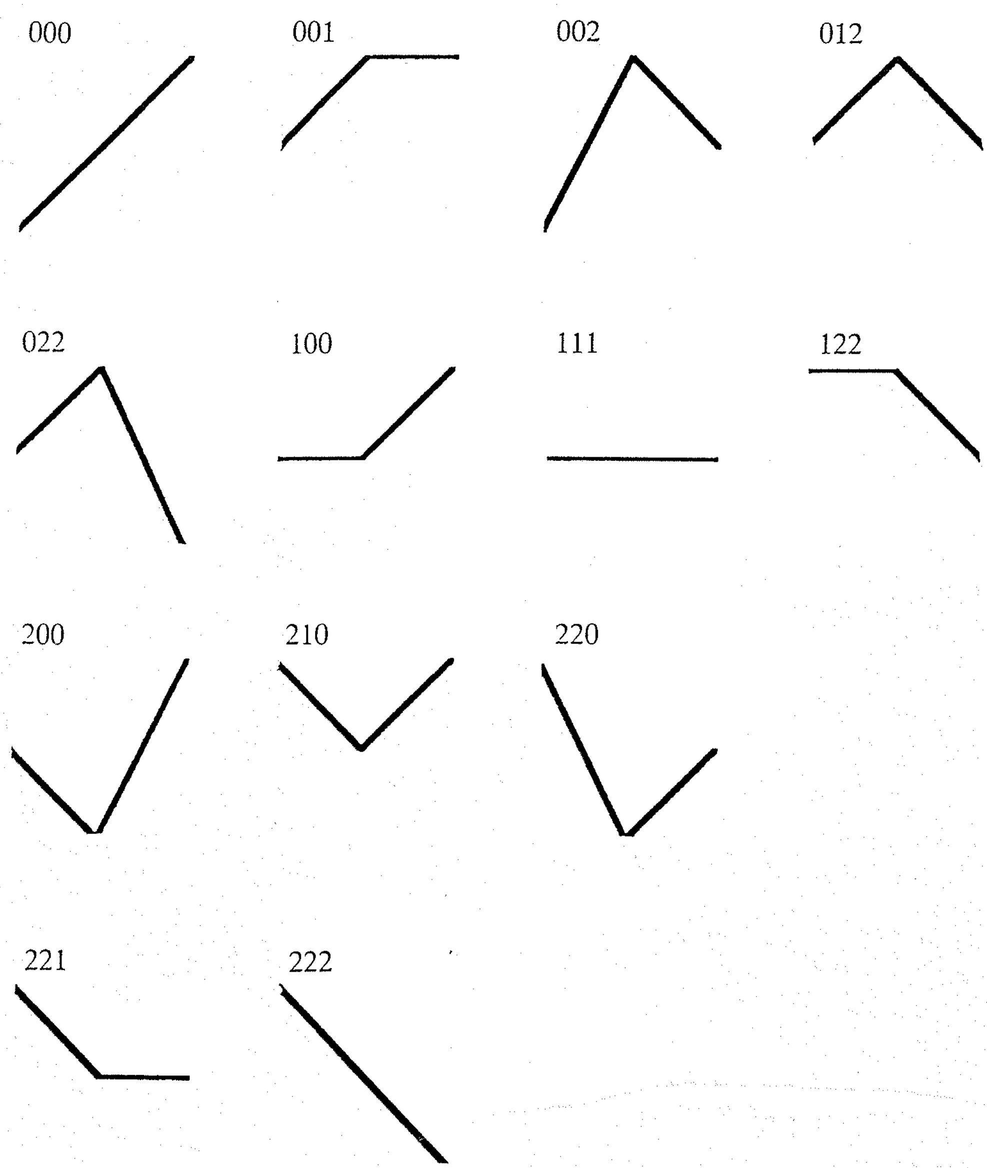
For morphologies of length three, assuming symmetry, the *13 possible ternary combinatorial contours* (CC_L) are[13]

000		222
001	100	221
002	111	220
012	122	221
022		222

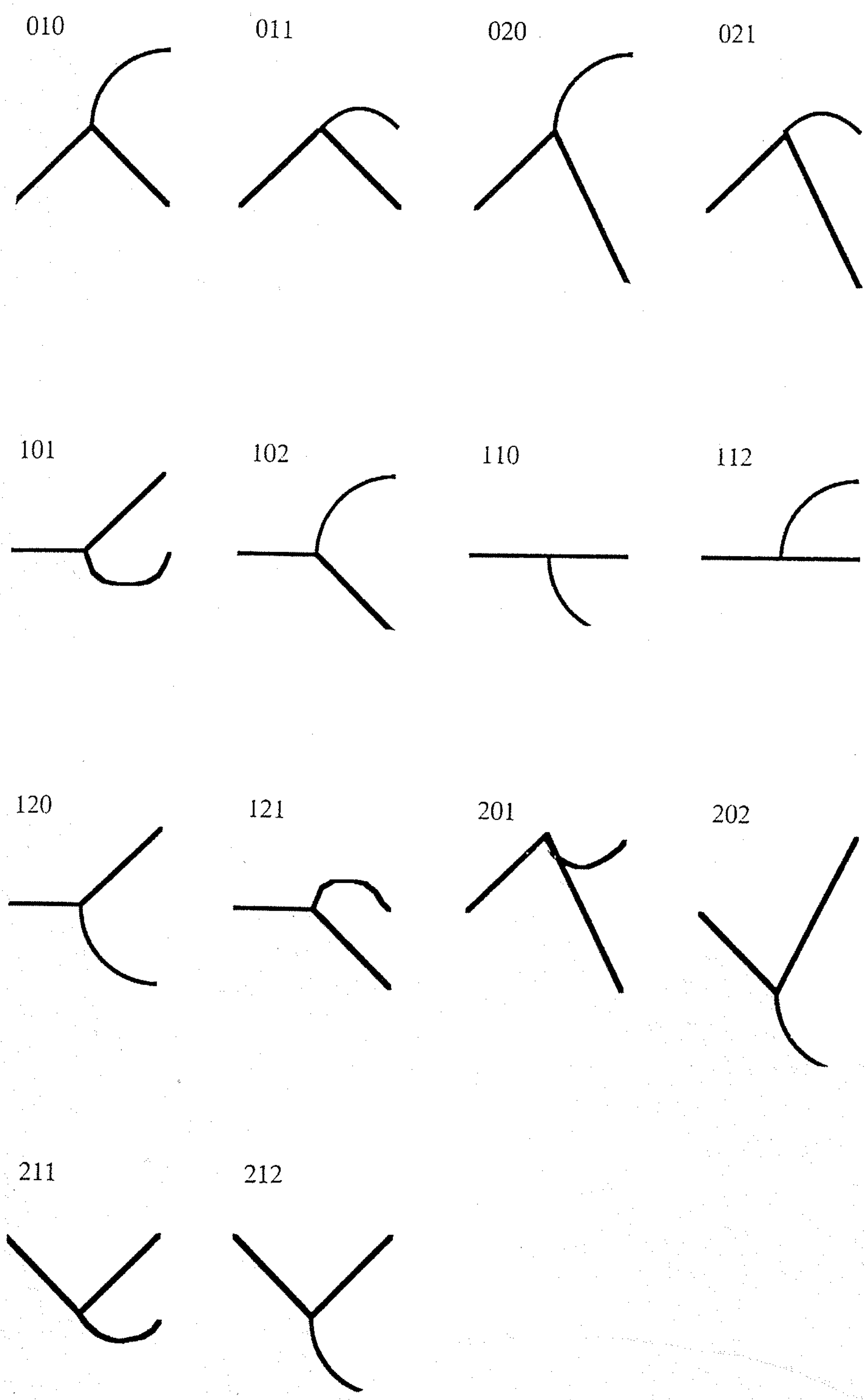
The 14 *impossible ternary combinatorial contours* are

	101	211
	102	212
021	110	202
020	112	201
011	121	
010	120	

The 13 possible ternary contours (L=3):



The 14 impossible ternary contours (L = 3):



For L>3, the lists become too long, even for the possible contours. The following chart shows the values for morphologies of length 6 and less.

Length	**#Descriptions**[14]	**#Possible(CC_L)**	**Ratio Possible**[15]
2	3	3	1
3	27	13	.481
4	729	75	.103
5	59,049	541	.009
6	14,348,907	4,683	.0003

The number of possible descriptions, since it is an exponential to the third power, gets "big very fast," as does the ratio of impossible to possible contours. The number of possible contours gets big much slower, and for musical purposes, is clearly the more significant number. For a morphology of a given length, it describes how many distinguishable "melodies" are possible—assuming that only contour relations of a very restricted type, ternary and symmetrical, are used for the distinction. However CC_L itself also gets big very fast.[16]

The number of possible three-valued contours (CC_L) can be expressed by the formula

$$\sum_{h=1}^{L} h!\, S(L,h)$$

where S(L,h) is a *Stirling number of the second kind.*[17] For example, for L=3, the formula can be expanded as follows:

$$\begin{aligned} & S(3,1) + 2!S(3,2) + 3!S(3,3) \\ = \;& 1 + (2*3) + (6*1) \\ = \;& 13 \end{aligned}$$

where the Stirling numbers of the second kind are S(3,1) = 1, S(3,2) = 3, and S(3,3) = 1.

For L=4 the equation is

$$\begin{aligned} & S(4,1) + 2!S(4,2) + 3!S(4,3) + 4!S(4,4) \\ = \;& 1 + (2*7) + (6*6) + (24*1) \\ = \;& 75 \end{aligned}$$

where the Stirling numbers of the second kind are S(4,1) = 1,S(4,2) = 7,S(4,3) = 6, and S(4,4) = 1.

Stirling numbers of the second kind are familiar from combinatorics and can be roughly expressed as

the number of ways to place L objects into h boxes with no box empty.

This naturally includes the possibility that each of the h boxes contains one or more of the L objects. This results from the fact that in a ternary contour, any or all of the values may be equal. A slightly different way of stating this problem is "how many ways can one partition a set of L elements into h non-empty, disjoint subsets" (Knuth, page 73).

This equation might also be thought of more in terms of *ranking*.[18] That is, each position in the contour has a rank of between {0, n−1}, but, importantly—and this invokes the use of the Stirling number formula above—these rankings do not *partition* the morphology; more than one position in the morphology may be of the same rank. In fact, since all the objects may be equal ("you can use an object as many times as you want"), the ranking might only employ one number (all the objects would be "0"). With 5 objects, two of which are equal, the highest ranking is 4. The question is then stated, in a slightly different way, "how many different rankings are possible with n objects, assuming that the only criteria for ranking is less than, equal to, or greater than."

The number CC_L is always less than L^L, and always greater than $L!$. CC_L gives the precise number of possible contours for L objects, where contour is defined in a strictly *ternary,* or three-valued way.

IV. Comments and Examples

Impossible contours form a kind of "negative space" for contours in general. They are only impossible, it should be pointed out, in a two-dimensional context because of the well-ordering aspect (see the proof below). But impossible contours can often be viewed as the conflicting superposition of two or more possible ones (this is easy to see from the diagrams where $L=3$). If multi-dimensional parameters are used, or a toric representation, it is easy to imagine how "impossible contour" illusions might be generated, like a morphological Shepard tone.

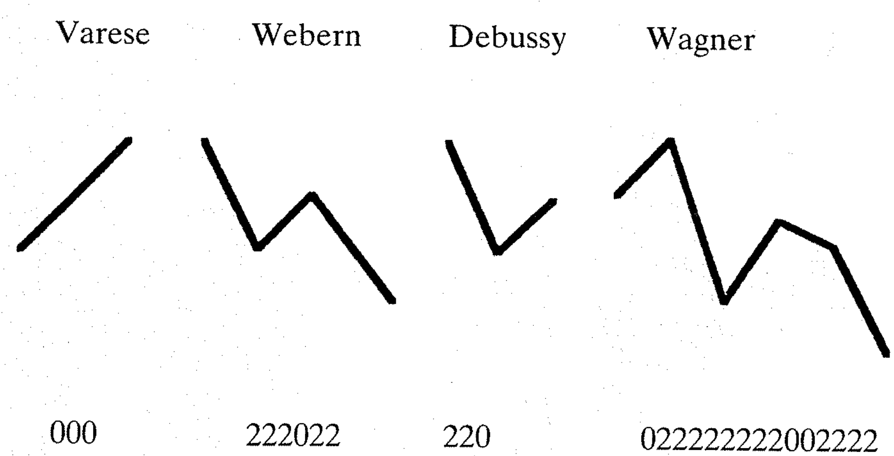
Compositional Uses

The notion of "possible and impossible contour" arose in one of the author's (Polansky) compositions, in a surprising way. Computer software that created melodies based on combinatorial contour alone was used to stochastically create melodies whose combinatorial contour similarity, or distance, was a certain predetermined value from some source melody (by some contour metric). A simple algorithm stochastically "dropped" random contour matrices and tested them (with any one of a number of combinatorial contour metrics) to see if they were within selected ranges of similarity to the source. With melodies of any length, most of the matrices created would not actually describe existing melodies, even though their metric values might qualify them for inclusion.[19]

Contour and Hierarchical Forms

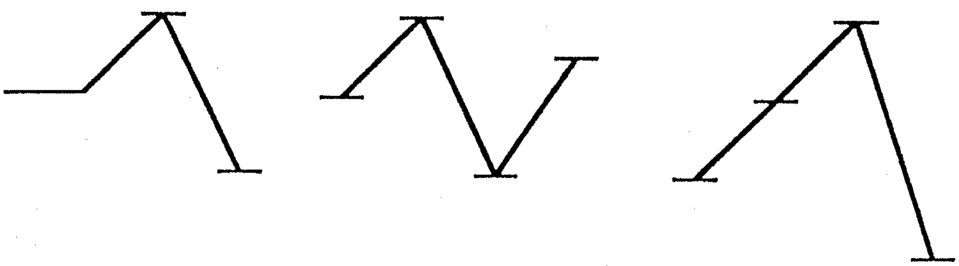
The *number of possible contours* tells us, in a quite restricted sense, how many "forms" there are. For example, if one considers the statistical profile of tripartite musical forms in some parameter (say the mean pitch, or temporal density), then there are surprisingly few large-scale forms (only 13). That is, the general pitch height (or temporal density, or loudness) means of the three sections of a piece can only have 13 possible configurations (in terms of their contour). There are fewer four-part forms than would have been expected—only 75! The number of possible contours can provide a kind of formal lexicon for large-scale analysis. We suspect that as larger and larger hierarchical groupings of musical events are made by listeners, contour will become more important in perceiving large-scale formal variation. We might be concerned that a given section of a piece was "more something" than another, or "more or less equal to," without being too particular as to *how much.* In this case, the specifics of the possibilities of contour become even more interesting.

Data from Tenney and Polansky (1980) and Polansky (1978) show parametric means of sections of works distinguished by the program described in those two articles. The four works analyzed by Tenney and Polansky were Debussy's *Syrinx,* Varese's *Density 21.5,* Webern's *Concerto, op. 24* (2nd Movement), and the English horn solo from Wagner's *Tristan and Isolde.* The large-scale pitch contours of these works are shown below.



These contours show the mean pitches of the highest hierarchical level of each piece (called *sections*).[20] Using the notion of ternary contours, we can simply describe each of these works as an n-digit number (admittedly within the very restricted domain of the computer algorithm used). Even though the two "runs" (using different weightings) for *Syrinx* produced different values, the contours are the same (220). These contours, while obviously involving a tremendous amount of information reduction, in some way describe the form of the piece, as would contours in other parameters, and raise interesting possibilities for formal comparisons and classifications.

Using data from Polansky's hierarchical analysis of Ruggles' *Portals,*[21] the contours of three different parametric means—pitch, intensity, and temporal density—at the highest hierarchical level can be compared, illustrating aspects of Ruggles' large-scale design in the structure of this work:



Pitch	Intensity	Temporal density
102022	020022	002022

Contour and Scale Theory

A more unusual application of contour comes from scale theory, particular of scales with a relatively small number of intervals and degrees, where interval width can be less important than the contour relationships of the interval widths themselves. That is, contour becomes important if one is not primarily interested in the exact widths of the intervals, but in whether one is wider, narrower, or (more or less) equal to another. This is the case, to some extent, for Central Javanese *slendro,* which has 5 distinct pitches (ignoring, for the moment, the stretched octaves), and 5 distinct adjacent intervals

(I–II, II–III, III–V, V–VI, VI–I′, summing usually to a 10–20-cent stretched octave). Because of the tremendous variations in different slendro tunings, foreign and Javanese scholars have often characterized different slendro tunings by calling their intervals *large, equal,* or *small* (where equal is roughly the five-tone equal 240-cent interval). This approach gives a surprisingly good depiction of the character of a given slendro tuning and is often used by Javanese musicians themselves, since the variation of a given interval is often (but not always) considered to be less important than a relationship between two intervals of "bigger than," "(more or less) equal to," or "smaller than." On a very simple level, this means that one can imagine the *number of possible slendro* as the number of possible combinatorial contours of length 5, or 541. Even if that number is simply the number of *types* of slendro (according to a very specific classification), with interval magnitude variations on each of these, one gets some general notion of the degree of diversity in the Central Javanese tuning tradition, and perhaps a basis for further analysis.

V. Proof

The formula above for CC_L, involving Stirling numbers of the second kind, is well known in combinatorics. It remains to be proven that the number of *three-valued contours* is in fact equal to the number of ways that "L objects can be placed in h boxes, so that none of the boxes is empty, and where any one of the L objects may be used several times"—in other words, the number of orderings corresponding to h different ranks.

The correspondence between the "number of possible contours" or rankings and the familiar combinatorial statement is not at all obvious, and requires proof. From a mathematical perspective, it clearly needs to be proved. From a musical standpoint, the authors' experience is that it is a difficult connection to explain in ordinary language.

The structure of the proof is as follows. Every morphology [M] can be written as a *reduced morphology* [*R(M)*][22] of a certain *height* [*h(M)*], where that height is always less than or equal to the *length* (L) of the morphology. The contour [*C(M)*] is then defined as the set of $(L^2-L)/2$ ternary relations between the values of a morphology. The central theorem is then proved that the number of possible C(M)'s is the same as the number of possible R(M)'s, meaning that the number of possible contours is established by the familiar equation involving Stirling numbers of the second kind.

Definition: A set V is *ordered* if there is a transitive relation "$<$" on V such that for every pair of distinct elements v_1 and v_2 in V, either $v_1<v_2$ or $v_2<v_1$.

Comment: This allows a more formal definition for a *morphology*. The set of integers, the set of durations, the set of steady-state timbres considered by amplitude of the 5th partial, the set of pitches, the set of positive integers, and the set of points in a line are all ordered. The set of points in the plane is not ordered, nor is a 2-dimensional perceptual representation of a timbre space with "brightness" and "attack length" as the two axes.

Definition: A *morphology* is a finite sequence of (not necessarily distinct) elements chosen from an ordered set. If that set is denoted V, we may represent a morphology of *length* L by $(v_1, v_2, \ldots, v_L)$, where the $v_k \varepsilon V$ are called the values (or elements) of the morphology.

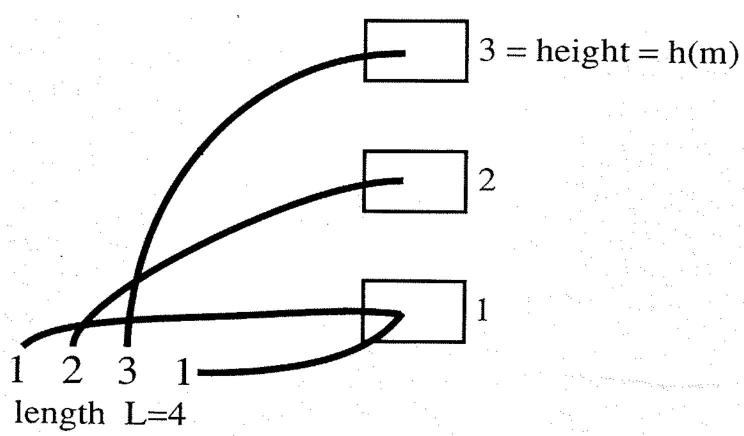
Comment: The sequence of pitches (C, F♯, G♯, C), a finite sequence of positive integers, or a set of four durations are all morphologies.

Definition: Let M be a morphology whose values are chosen from a set V. The *reduction* of M, denoted *R(M)*, is a morphology with positive integer values constructed as follows. Let v be the smallest value appearing in the morphology; replace very occurrence of v with 1. Let v′ be the next smallest (if any) appearing in the morphology; replace very occurrence of v′ with 2. Continue in this fashion until all values in the morphology are exhausted. The *height h(M)* of the morphology is the largest integer appearing in R(M); it follows that every integer from 1 to h(M) appears in R(M).[23]

Comment: For example, for M = (C, E, G, C), we have R(M) = (1,2,3,1), and h(M) = 3.

Example. Construction of R(M) with height h from a morphology

Ordered set = {C, C♯, D, D♯, ..., B} or {0, 1, 2, 3, ... 11}
Morphology M = {C, E, G, C} or {0, 4, 7, 0}
Reduced morphology R(M) = {1, 2, 3, 1}



Definition: The *combinatorial contour C(M)* of a morphology of length L is a sequence of $L(L-1)/2$ values chosen from (0,1,2)[24] as follows. The symbols {0,1,2} are used to indicate the relationship between ordered pairs of values v_i and v_j in a morphology: if $i<j$, so that the value v_i appears before v_j in the morphology, then:

0 indicates $v_i<v_j$,
1 indicates $v_i = v_j$, and
2 indicates $v_i>v_j$[25]

The first $L-1$ values of C(M) represent the comparison of consecutive values of M with all those that follow it; the next $L-2$ represent the comparison of the second value of M with all those that follow it; the next $L-3$ represent the comparison of the third value of M with all those that follow it, and so on. This is simply a formal description of the construction of the ternary combinatorial contour number described earlier.

Comment: For M = (C,E,G,C), we have C<E, and C<G, and C = C, so the first three values of C(M) are 0, 0, 1. Since E<G and E>C, the next two values of C(M) are 0, 2. Finally, since G>C, the final value of C(M) is 2. Hence C(M) = (001022).

Theorem: For morphologies M and M′, we have R(M) = R(M′) if an only if C(M) = C(M′).

Proof: Since the reduction of a morphology preserves the length of the morphology and all the order relations between the values of the original morphology, if R(M) = R(M′), then C(M) = C(M′).

Now suppose C(M) = C(M′). First we prove that h(M) = h(M′). If this were not the case, suppose M is the morphology with the larger height, that is, h(M)>h(M′). Choose a sequence of h(M) values v_{k_1}, $v_{k_2}, \ldots, v_{k_{h(M)}}$ from M which correspond to 1, 2, ..., h(M) in R(M); note that these values do not necessarily appear in this order in M, that is, although $v_{k_1}<v_{k_2}< \ldots <v_{k_{h(m)}}$, the subscript k1 is not necessarily smaller than k2, etc. Since the contour C(M) contains 0's and 2's corresponding to the comparisons between these values, the contour C(M′) must also. But then we can construct a sequence of h(M) distinct values in the morphology M′, contradicting h(M)>h(M′).

Now let $R(M) = (v_1, v_2, \ldots, v_L)$ and $R(M') = (v'_1, v'_2, \ldots, v'_L)$ and suppose that $R(M) \neq R(M')$. Since their values must differ in some place, say $v_k \neq v'_k$, choose M to be the morphology of the pair such that $v_k<v'_k$. Further, let k0 be the (not necessarily unique) position of the largest value in R(M′) that is greater than its corresponding value in R(M); that is, $v_{k0}<v'_{k0}$, and v'_{k0} is the largest for which this inequality is true. Since $v'_{k0} \leq h(M') = h(M)$, there is a k1 such that $v_{k1} = v'_{k0}$; thus, $v_{k1}>v_{k0}$. Since C(M) = C(M′) and $v_{k1}>v_{k0}$, then $v'_{k1}>v'_{k0}$. But $v'_{k0} = v_{k1}$, so $v'_{k1}>v_{k1}$, contradicting the maximality of v'_{k0}. *QED*

The following diagram shows the general structure of the above proof. The three steps show the greater than (above), equal to (on the same line as) and less than (below) relationships of the elements in the two morphologies M and M′, with C(M) = C(M′).

Diagram of the proof

STEP 1			STEP 2			STEP 3	
$(v_{k0} < v'_{k0})$			$(v_{k0} < v'_{k0} = v_{k1})$			$(v_{k0} < v'_{k0} = v_{k1} < v'_{k1})$	
M	**M′**		**M**	**M′**		**M**	**M′**
	v'_{k0}	→	v_{k1}	v'_{k0}	→		v'_{k1}
v_{k0}			v_{k0}			v_{k1}	v'_{k0}
						v_{k0}	

(Since C(M′) = C(M), the relationship between v'_{k_1} and v'_{k_0} must be the same as between v_{k_1} and v_{k_0}, which contradicts the maximality of v'_{k_0} among v'_k which are greater than their corresponding v_k.)

Corollary: The number of contours C(M) for morphologies of length L is the same as the number of reduced morphologies of length L.

The next theorem uses the formula given above for the calculation of CC_L. In the formula below

Theorem: The *number of reduced morphologies* R(M) of length L is

$$\sum_{h=1}^{L} h!\, S(L,h)$$

Proof: We have $h(M) \leq L$. A reduced morphology of height h is determined by giving each of the L positions in the sequence a value from 1 to h, making sure that each value from 1 to h is taken at least once.[26] *QED*

VI. Further Problems

The formal definition of possible and impossible contours raises several theoretical and compositional questions.

Generalization to n-ary Contours

Important theoretical and mathematical questions remain, mostly pertaining to ways of generalizing these concepts. First, how may the formula above be extended to other than ternary contours, or *n-ary* contours? For example, if quintary values are allowed, as in: *a lot less than, less than, equal to, greater than, and a lot greater than,* how many possible contours are there?[27] In this case, we would have to investigate the *possibility* of the following number of contour descriptions:

$$5^{[(L^2-L)/2]}$$

With a generalized n-ary contour, the notion of "between-ness," essential to the mathematical discussion of contour, becomes more complex if the situation is not limited to ternary contours. Because of this, the investigation of transitivity and the resulting equations for *the number of possible n-ary contours* is even more complex and, as far as we are concerned, still an unsolved question (at least in the music theory literature).

Michael Friedmann posed the very interesting question: are all binary contours possible?[28] This question may be slightly restated so that contours with adjacent equal elements are excluded from the discussion. In other words, "are all binary contours without equality possible?" The answer is no. The $L=3$ ternary contours **202** and **020** (inversions of each other) are counter examples: neither includes an equality, but both violate transitivity. For $L>3$ there will be many more impossible contours that do not include an equality.

Contour and Scale

The discussion of n-ary contours can be generalized further, so it is no longer a question of contour, but perhaps of *scale:* the number of possible relationships defines the scale, or gamut, in a given perceptual domain. This was mentioned earlier as the "grain" of distinction. The base of the binomial coefficient exponent is the number of degrees of difference we are willing to consider, or the possible ways any two elements in the morphology may be related. For example, the second order binomial coefficient raised to the 12th power is the number of possible "contours" of 12 values, or in other words, the number of possible rankings of length twelve chosen from 12 pitches (with of course, the possibility of equality). This seems to suggest a theoretical continuum between the mathematics of contour and the mathematics of other aspects of atonal set theory.[29] A more general and elegant statement of this generalization is an interesting and challenging problem for music theory.

Asymmetry

The issue becomes even more complicated and interesting if asymmetrical contours are considered, as in *less than, equal, greater than,* *a lot greater than.* The mathematics of asymmetrical contours, with a different set of logical possibilities, are, we suspect, different from the mathematics of symmetrical ones. This situation also presents an area for further research, in terms of both the mathematics of the situation and the investigation of its musical applications.

NOTES

1. Even authors like Friedmann, Marvin and Laprade, and Polansky, who work with "combinatorial" definitions of contour, include linear contours as a "less-sensitive" first step.
2. Friedmann, in his work on contour in Schoenberg, distinguishes between what I have called "linear and combinatorial" contour by his two constructs CC and CAS: "(CC) describes contour relations among all the pitches—not merely the adjacent ones as the CAS does—and can reflect the occurrence of pitch repetitions."
3. On a philosophical level, we might say that the elements in our morphologies, with their associated contour relationships, need to be *distinguishable* in some dimension, but not necessarily the one for which we are forming the contour description. This would naturally include the possibility of equal values.
4. The *sgn* function is defined as sgn(a,b) = 1, if $a>b$; 0, if $a=b$; -1 if $a<b$. Note that it is *not* a metric, because sgn(a,b) does not equal sgn(b,a).
5. In the theoretical literature our term "morphology" is usually rendered as "ordered set."
6. Marvin and Laprade state (p. 231) that their COM-matrix, which is a combinatorial matrix functionally equivalent to our ternary number contour description, "furnishes a much more complete picture since it is not limited simply to relationships between adjacent contour pitches." Polansky has found, however, in both compositional and analytical applications, that although linear contour metrics are "less sensitive" to the "complete picture" of contour, they are highly useful in modelling many perceptual situations and, in fact, give very different results from combinatorial metrics. That is to say, they are not "better or worse" than combinatorial measures, simply different.
7. These ideas become extremely important if one is interested in contour *similarity,* or what Polansky has called *contour metrics.* Several authors (Marvin and Laprade, with their CSIM(A,B), and Polansky, with his OCD, OLD, and other classes of metrics) have developed precise combinatorial similarity functions, although as usual the notation is very different. For example, Marvin and Laprade's CSIM(A,B) returns a value of "1" when two contours are equivalent, where Polansky's metrics (like the OCD, which in one of its particular forms is equivalent to CSIM(A,B)) naturally return 0 (by the definition of a metric).

   A further theoretical generalization of these metrics has been implemented and published in software form, written in the computer music language HMSL. It includes more general definitions of adjacency (for example, a concept of adjacency that shifts during the measure), interval (including intervals to elements not strictly in the set or morphology), and even degree of combinatoriality.
8. One needs to be cautious when exploring new applications of binomial coefficients, for according to Knuth, "there are so many relations present that when someone finds a new identity, there aren't many people who get excited about it any more, except for the discoverer." We hope the musical applications have not been so completely exhausted!
9. This particular procedure is equivalent to Marvin and Laprade's way of constructing the matrix from the INT_n function.

10. One very elegant one was suggested to us by David Lewin, and Polansky and Phil Burk formulated another one in the design of the numerical proof of the number of possible contours (see note 16 below). A graduate student at Dartmouth, Gerald Beauregard, has also devised an ingenious algorithm that uses directed graphs.
11. A nice example of this was suggested to us by theorist Stephen Haflich and explained to us by Polansky's six-year-old friend Nicolas Collins Weiler: in the children's game "rock, paper, scissors," scissors cuts paper, paper covers rock, and rock crushes scissors.
12. In fact, one such algorithm was designed by Phil Burk in the construction of the numerical proof of the number of possible contours up to $L=6$.
13. These thirteen possible ternary contours form a kind of superset of many of the existing contour lists in the literature, like Seeger's and Adams'. For example, one of Adams' typologies (p. 199) does not include three-element contours with equal elements or contours that are completely ascending or descending (these are subsumed logically under his two element contours), leaving only (in my notation) {200, 210, 220, 002, 012, 022}, which he calls

$$\{D_1R_1S_1, D_1R_1S_2, D_1R_1S_3, D_1R_2S_1, D_1R_2S_2, D_1R_2S_3\}.$$

    Other systems eliminate reflections, inversions, and so on. Since we are not invoking any direct reference to particular pre-existing musical styles or theories, there is no reason to exclude any of these contours.
14. The number of descriptions is 3^{L_m} or:

$$3[(L^2-L)/2]$$

15. #Descriptions/#Possible, rounded off to the last digit.
16. The table is a result of a numerical proof (by Polansky and Phil Burk, in HMSL) of the number of possible three-valued contours for melodies of short length, a proof equivalent to the non-numerical one given at the end of this paper, but only for lengths 1–6. A home computer was not fast enough to easily calculate possible contours for numbers greater than 6 in a reasonable amount of time. Using Stirling numbers, the reader can easily work out this simple formula for higher numbers.

    Coincidentally, Marvin and Laprade, in their listing of C-Space Segment Classes, stop at cardinality 6 as well. Even with the reduced numbers resulting from their equivalence classes, the numbers get big quite fast.
17. Knuth, p. 66. The formula for computing a Stirling number of the second kind, although it is more easily determined using a standard table, is:

$$\sum_{k=1}^{h} \frac{(-1)^{k-h}k^L}{k!\,(h-k)!}$$

    For an excellent discussion of the combinatorial applications of Stirling numbers, along with some fundamental derivations and identities, see Grimaldi. Our thanks to composer-theorist Robert Morris for directing us to that important reference.
18. Functionally, exactly the same as Morris' pitches "numbered in order from low to high, beginning with 0 up to $n-1$." This concept is also used by Marvin and Laprade, and Friedmann.

19. This occurred in the composing of a piece entitled *17 Simple Melodies of the Same Length,* a work which "listened" in real time to a number of melodies improvised by a live performer and then sorted them according to their combinatorial contour, using a metric documented in Polansky, 1987. When this stochastic method was used, in pieces like *Distance Musics* (first performed in 1986; Polansky, 1987a) and *Bedhaya Sadra/Bedhaya Guthrie* (Polansky, 1990), generating morphologies with the required combinatorial contour proved too difficult, owing to the overwhelmingly large number of possible matrix descriptions as compared to the number of possible contours. Eventually, *mutation functions* that generate a morphology of a given metric value to another were designed in HMSL and used in several works.

    There are computer algorithms for deciding on the "impossibility" of combinatorial contour descriptions; these algorithms were used (by Polansky and Phil Burk) to generate a numerical proof (up to L=6) of the theorem proved in this paper. These algorithms are too slow to work in a real-time performance context and are an interesting problem for further investigation: "From a given ternary contour description, how does one know if it is impossible or possible?" It is possible that a new notation, other than the long ternary number representation of a matrix, will be needed for this particular test.

20. The following table gives the actual pitch means of the highest level sections for the above pieces (the two different runs for *Syrinx* are shown):

	Syrinx	*Concerto*	*Density*	*Tristan*
Section 1:	17.19/20.33	38.23	13.31	16.03
Section 2:	5.07/8.0	32.90	19.06	16.56
Section 3:	12.35/12.35	37.83	20.67	13.50
Section 4:		22.63		15.03
Section 5:				14.32
Section 6:				7.79

21. The parametric means for the four sections of *Portals* distinguished by the program are:

	Pitch	**Intensity**	**Temporal Density**
Section 1:	29.28	5.22	3.38
Section 2:	29.27	5.47	3.54
Section 3:	31.36	5.11	3.6
Section 4:	28.61	5.3	3.3

    Note that the question of the *grain of contour* arises with even this simple example: the pitch means of Sections 1 and 2 are considered to be "close enough" here to be equal, a choice we made more or less arbitrarily in drawing these contours. In all other cases, the values were *significantly different,* that is, by some percentage of the range of values that might occur, thus permitting a contour distinction.

22. Our R(M) is (more or less) functionally equivalent to Friedmann's *CC* and Marvin and Laprade's *Cseg.*
23. This is of course similar to the way in which Morris constructs his normal contour forms, or Friedmann his "CC".

24. Knuth suggests a more typographically sound notation than (−1,0,1) and less numerically "loaded" than (0,1,2) that uses underlines: $(1,0,\underline{1})$. (−1,0,1) and $(1,0,\underline{1})$ are examples of what Knuth calls a "balanced number system," because they are symmetrical with 0 in the middle. We use the ordinary ("unbalanced") ternary notation (0,1,2) in describing lists of contours because of the ease of counting and devising simple computational aids in the ternary numbers.
25. This is simply the *sgn* function, or a formal statement of the usual way of notating three-valued contour.
26. Knuth, p. 73.
27. It is quite easy to make the notion of n-ary contour consistent with, or at least a natural extension of, the literature on contour. For example, Marvin and Laprade (p. 228) state: "The decision not to define the intervallic distance between c-pitches [their notation for the ranked values in an ordered set] reflects a listener's ability to determine that one c-pitch is *higher than, lower than, or the same as another* [emphasis mine], but not to quantify how much higher or lower." If we simply insert "a lot higher than" and "a lot lower than" into their list, quintary contours would seem to fit just as well into their work with c-segments and equivalence classes.
28. Personal communication.
29. Robert Morris, for example, points out a similar "impossibility" problem in the Forte interval vector (personal communication).

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REVIEWS

Ear Training for Twentieth-Century Music
by Michael L. Friedmann
New Haven: Yale University Press, 1990
xxvi, 211 pp.

REVIEWER
Cynthia Folio

This book treats a subject that many in the field of music theory have long awaited—ear training for twentieth-century music. It reflects a growing concern in the recent theoretical literature with perception and the ability to hear musical structure. The book combines numerous exercises and musical examples—all approached aurally—with one of the clearest explanations available of the concepts and terms used in atonal analysis. Musical examples include excerpts from standard contemporary repertoire and melodies carefully written by the author, all exemplifying specific goals in hearing structures. The exercises serve as an excellent springboard into the aural and visual analysis of entire works, leaving it to the teacher or reader to apply these detailed relationships to larger formal structures.[1]

But *Ear Training* is more than a pedagogical text and it encompasses more than ear training, successfully incorporating theoretical material as well. Most impressive is the author's presentation of much more than the techniques of identifying isolated harmonies or melodic fragments; his aim is hearing *structure*—an admirable goal and one he convinces us is attainable through study and practice. Many of the exercises are based on his "conviction that the precise articulation of structural relations provides the basic foundation for twentieth-century music, and that perceiving them is a precondition for understanding affective content and gesture" (p. xxiii). His definition of a "good ear" is "one that perceives and retains musical structures and understands their role in a musical *transformation* or other compositional process" (p. xxiii, italics in original).

The book is organized so that it progresses from simple concepts to the more difficult. The first chapter, "Calisthenics," conditions the reader for the following chapters. The next chapter introduces dyads and emphasizes the various ways of hearing an interval, either in pitch space or pitch-class space. Chapter three, "Processes: Pitch, Pitch Class, and Contour Relations," immediately takes the reader beyond intervals into hearing "processes" like retrograde, transposition, inversion, invariance, and contour relationships. Chapters four, five, and six describe the characteristics and structural properties of all the trichords and tetrachords, and selected sets of more than four elements. Appendix I contains numerous musical examples from the works of Debussy, Bartok, Stravinsky, and Schoenberg to be used as a resource for dictation, sight singing, and identification of structures and processes. Appendix II contains additional exercises and is followed by a glossary and an index.

One important theme that runs through the various chapters is the emphasis on hearing *processes:*

> In grappling with twentieth-century music we cannot confine ourselves to the definition of musical *structures,* such as interval types or set classes. It is equally important to perceive a range of *relations,* or transformations, that can connect different structures. Relations of this sort can be conceptualized as operations or processes, three of which are retrogression, transposition, and inversion. These processes treat musical elements as combinations either of pitches or pitch classes. (p. 23)

The act of hearing interval equivalencies in the chapter on "Dyads" prepares the reader for the analysis of more complex relationships. The author carefully distinguishes between pitch space and pitch-class space, thereby illustrating different ways of hearing and analyzing intervals. For example, students are asked to compare interval pairs on the basis of (a) ordered pitch interval; (b) unordered pitch interval; (c) ordered pitch interval mod 12; (d) unordered pitch interval mod 12; or (e) unordered pitch class interval (p. 19).

After defining and hearing equivalencies, we progress to definitions of key processes, complete with exercises for hearing various transformations. The first transformations introduced are transposition and inversion, along with invariance, symmetry and index number. Pitch space and pitch-class space are always treated separately. For example, $T^{p}_{+/-n}$ signifies pitch transposition by n half-steps, upward (+) or downward (−); whereas T_n is pitch class transposition by n where n is greater than or equal to 1 and less than or equal to 11 (pp. 24–25).

Included in the author's list of processes is an imaginative approach to hearing contour and contour identities.[2] Again, it reflects the author's emphasis on hearing relationships: