6 Scales, modes, and systems

THE FORMATION OF heptatonic scales from tetrachords was mentioned briefly in chapters 1 and 5. In the present chapter, scale construction will be examined at greater length—in particular, the formation of non-traditional and non-heptatonic scales from tetrachordal modules. Before introducing this new material, however, a brief review of the salient features of the Greek theoretical system is necessary as an introduction to scale construction.

The hierarchy of scalar formations

The ancient Greek theorists recognized a hierarchy of increasingly large scalar formations: tetrachord, pentachord, hexachord, heptachord, octachord, and system. The canonical forms of each of these scalar formations may be seen in 6-1. The smaller formations were finally absorbed into the Perfect Immutable System which with its fifteen pitch keys or tonoi was the highest structural level of the Greek theoretical doctrine. As the tetrachordal level has been introduced in earlier chapters, the discussion will focus on the pentachord and larger structures.

The pentachord

Pentachords may be considered as tetrachords with disjunctive tones added at either extremity. They divide the perfect fifth into four subintervals and occur in several forms in the various modes of heptatonic scales. The two forms of greatest theoretical importance are described in 6-1. While of relatively minor musical prominence, the pentachord has considerable pedagogical value in explaining how certain tunings and scales may have arisen.

For example, Archytas's complex septimal tuning system can be best understood by considering not just the three species of tetrachord, but the pentachords formed with the note a whole tone below. These would be the note hyperhypate for the meson tetrachord and mese for the diezeugmenon (Winnington-Ingram 1932; Erickson 1965). By the use of the harmonic mean between hyperhypate (8/9) and mese (4/3), Archytas defined his enharmonic lichanos as 16/15. His tuning for the note parhypate (28/27) in all three genera was placed as the arithmetic mean between the 8/9 and 32/27, the diatonic lichanos. This construction may be seen in 6-2.

The notes D F G and A form the harmonic series 6:7:8:9 and the notes D G𝄫 A a minor triad, 10:12:15. The 7/6 which the hyperhypate (D) makes with parhypate (F) is found in all three of his genera and is duplicated a fifth higher between mese (A) and trite (C). This interval was very important in Greek theory and had its own name, ekbole (Steinmayer 1985). It occurs in the Dorian harmonia shown in 6-4 and in the fragments of surviving Greek music.

As this interval has the value of 7/6 only in Archytas's tunings and those others of the 7/6 pentachordal family (chapter 4), it is interesting to consider analogous pentachords with the 28/27 replaced by other intervals. 6-2 also depicts such a system, employing a more Aristoxenian 1/4-tone interval, 40/39, which was used by the theorists Eratosthenes, Avicenna, and Barbour in their genera (See the Main Catalog and 4-3). This system has a number of interesting harmonic and melodic intervals and could be played very well in 24-tone equal temperament.

Miscellaneous pentachordal structures

According to Xenakis, chains of conjunct tetrachords and pentachords (trochos) are used in the liturgical music of the Greek Orthodox church

6-1. The hierarchy of scalar formations. The tetrachord may be any of the those listed in chapter 9. The interval of equivalence is the 4/3. The two canonical forms of the pentachord are given. Other forms occur in the various modes of heptatonic scales of different genera and may have the 9/8 interpolated between the tetrachordal intervals. With the addition of the octave 2/1, the heptachord becomes the Mixolydian mode of the complete heptatonic or octachordal scale. If the 8/9 is added below the 1/1 the scale becomes the Hypodorian mode transposed downwards by a whole tone (9/8). The next highest structural level is that of a system which contains all the lower ones. The octachord is the heptatonic Dorian mode.

FORM		NOTES
TETRACHORD		1/1 *a* *b* 4/3
PENTACHORD	1:	1/1 *a* *b* 4/3 3/2
	2:	8/9 1/1 *a* *b* 4/3
HEXACHORD	1:	1/1 *a* *b* 4/3 3/2 3*b*/2
	2:	1/1 *a* *b* 4/3 3/2 3*a*/2
HEPTACHORD		1/1 *a* *b* 4/3 4*a*/3 4*b*/3 16/9
OCTACHORD		1/1 *a* *b* 4/3 3/2 3*a*/2 3*b*/2 2/1

6-2. *Pentachordal systems.*

ARCHYTAS'S SYSTEM

D	E	F	G𝄫	G♭	G	A
8/9	1/1	28/27	16/15	9/8	32/27	4/3
6/5			5/4			
7/6		9/7				
7/6		8/7				

40/39 SYSTEM

D	E	F	G𝄫	G♭	G	A
8/9	1/1	40/39	16/15	10/9	32/27	4/3
6/5			5/4			
15/13		13/10				
5/4				6/5		
15/13		52/45				

(Xenakis 1971, and chapters 2 and 5). These chains exhibit cyclic permutation of their constituent intervals. Most importantly, they are examples of those rare musical systems in which the octave is not the modulus or interval of equivalence.

Additionally, more traditional heptatonic modes (echoi), some of which appear to have genetic continuity with classic Greek theory, if not practice, are employed. These may be analyzed either as composed of two tetrachords or as as combinations of tetrachord and pentachords. A number of tetrachords from these modes are listed in the Catalogs.

Some irregular species of Greek and Islamic origin are also listed in chapter 8 along with Kathleen Schlesinger's harmoniai to which they bear some resemblance. These divide the fourth into four parts and the fifth into five. The Greek forms are merely didactic patterns taken from Aristoxenos and interpreted by Kathleen Schlesinger as support for her theories, while the Islamic scales were apparently modes used in actual music. 8- or 9-tone pseudo-tetrachordal octave scales may be formed by combining these with appropriate fifths or fourths.

The hexachord, heptachord, and gapped scales

The hexachord and heptachord generally appear as transitional forms between the single tetrachord and the complete heptatonic scale or octachord. The hexachord appears as a stage in the evolution of the enharmonic genus from a semitonal pentatonic scale similar to that of the modern Japanese koto to the complete heptatonic octave. This 5-note scale is often called the enharmonic of Olympos (6-3) after the legendary musician who was credited with its discovery by Plutarch (Perrett 1926). This and other pentatonic scales may be construed as two trichords combined with a whole tone to complete the octave. The two intervals of the trichord may be a semitone with a major third, a whole tone with a minor third, or any other combination of two intervals whose sum equals a perfect fourth.

At some point the semitone in the lower trichord was divided into two dieses. This produced the spondeion or libation mode which consisted of a lower enharmonic tetrachord combined by disjunction with an upper trichord consisting of a semitone and a major third (6-3). This hexachord or hexatonic scale evolved into the spondeiakos or spondeiazon tropos. Eventually the semitone in the upper trichord was also split and a hep-

6-3. *Gapped or irregular scales. The notation used here reproduces that of the references. The plus sign indicates a tone 1/4-tone higher than normal. Unless otherwise noted, no particular tuning is assumed, but either Pythagorean or Archytas's supplemented as required with undecimal ratios would be appropriate historically.*

Pentatonic forms

ENHARMONIC OF OLYMPOS

e f a b c (e')

SPONDEION (WINNINGTON–INGRAM 1928)

e f a b c+ *or* e f+ a b c+

1/1 12/11 4/3 3/2 18/11 (2/1)

SPONDEION (HENDERSON 1942)

f a b d# e+ *or* e e+ f a b

SPONDEION (MOUNTFORD 1923)

1/1 28/27 4/3 3/2 18/11 (2/1)

Hexatonic forms

SPONDEIAKOS *or* SPONDEIAZON TROPOS

(WINNINGTON–INGRAM 1928)

e e+ f a b c

with b+ d' *&* c' *in the accompaniment*

DIATONIC OF WEIL & REINACH

(WINNINGTON–INGRAM 1928)

e f g a b d

with b♮ c & e' *in the accompaniment*

GAPPED SCALE OF TERPANDER & NICOMACHOS

(HELMHOLTZ 1877, 266)

e f g a b d (e')

DIATONIC OF GREIF

(WINNINGTON–INGRAM 1928)

d e f a b♮ c# (d')

SCHLESINGER (1939, 183)

1/1 11/10 11/9 11/8 11/7 1/6 (2/1)

Heptatonic form

CONJUNCT HEPTACHORD

c f g a b♮ c d

tatonic scale in the enharmonic genus resulted. This transformation may have been completed about the time of Plato, who writes as if he distrusted these innovations. In later times, the ancient pentatonic and hexatonic melodic patterns were retained in compositions for voice and accompaniment (Winnington-Ingram 1936).

In principle, a hexachord can be obtained from a heptatonic scale in four ways by omitting one tone in either tetrachord. 6-3 lists the versions found in the literature. In these cases, the omitted note is the sixth degree, though the second version which lacks the seventh instead is a plausible interpretation in some cases. Schlesinger's version is based on her theories which are described in detail in chapter 8.

Some controversy, however, exists in the literature about the tuning of these early gapped or transilient scales. The arguments over the relative merits of enharmonic or diatonic tunings were discussed by Winnington-Ingram (1928) whose scales and notation are reproduced in 6-3. Notable are his and Mountford's undecimal or 11-limit tunings for the pentatonic forms. Winnington-Ingram's undecimal neutral third pentatonic could be the progenitor of the hemiolic chromatic genus (75 + 75 + 350 cents) and diatonics similar to the equable diatonic such as 150 + 150 + 200 cents. Henderson (1942) has also offered two quite different non-standard interpretations of the enharmonic pentatonic based on etymological considerations.

The hypothetical diatonic versions of these scales according to the suggestions of several scholars are listed in this table as well. Weil and Reinach provide a conventional diatonic form (Winnington-Ingram 1928). The version of Greif appears to be derived from the Lesser Perfect or Conjunct System with the addition of a tone below the tonic as seen in the Dorian harmonia of 6-4 (ibid.). It should be compared with the ancient non-octaval heptachord which may also be formally derived from the conjunct system (6-1).

The medieval diatonic hexachord of Guido D'Arezzo, c d e f g a c', may be included with these scales too, although it is much later in time. In just intonation, it is usually considered to have the ratios 1/1 9/8 5/4 4/3 3/2 5/3, derived from the Lydian mode of Ptolemy's syntonic diatonic instead of the Pythagorean 1/1 9/8 81/64 4/3 3/2 27/16. In the septimal diatonic tuning of Archytas it would have the ratios 1/1 8/7 9/7 4/3 32/21 12/7.

6-4. The oldest harmoniai in three genera.

Dorian

ENHARMONIC d e f- g♭♭ a b c- d'♭♭ e'
CHROMATIC d e f g♭ a b c d'♭ e'
DIATONIC d e f g a b c d' e'

Phrygian

ENHARMONIC d e f- g♭♭ a b c- d'♭♭ d'
CHROMATIC d e f g♭ a b c d'♭ d'
DIATONIC d e f g a b c d'

Lydian

ENHARMONIC f- g♭♭ a b c- d'♭♭ e' f-'
CHROMATIC f g♭ a b c d'♭ e' f'
DIATONIC f g a b c d' e' f'

Mixolydian

ENHARMONIC B c- d♭♭ d e f- g♭♭ b
CHROMATIC B c d♭ d e f g♭ b
DIATONIC B c d e f (g) (a) b

Syntonolydian

ENHARMONIC B C- d♭♭ e g
CHROMATIC B C d♭ e g
DIATONIC c d e f g
2ND DIATONIC B C d e g

Ionian (Iastian)

ENHARMONIC B C- d♭♭ e g a
CHROMATIC B C d♭ e g a
DIATONIC c e f g a
2ND DIATONIC B C d e g a

The octachord or complete heptatonic scale

The union of a tetrachord and a pentachord creates an octachord or complete heptatonic scale. There is evidence, however, that initially two diatonic tetrachords were combined by conjunction, with a shared note between them, to form a 7-note scale less than an octave in span (6-1). The later addition of a whole tone at the top, bottom, or middle separating the two tetrachords, completed the octave gamut. Traces of this early heptachord may be seen in the construction of the Lesser Perfect System and in the irregular scales of 6-3 and 6-4.

Similarly, two enharmonic tetrachords were joined by disjunction with the 9/8 tone between them to create the Dorian harmonia to which a lower tone was added (6-4). An alternative genesis would connect two pentachords whose extra tones were at their bases to produce the 9-tone Dorian harmonia to which other tones might accrete. By analogy, both the enharmonic and diatonic proto-scales converged to the same multi-octave structures later called by the name of system. In the fifth century BCE the wide ditone or major third of the enharmonic genus was gradually narrowed to a minor or subminor third by a process termed "sweetening." Eventually, this process resulted in the chromatic genus which was raised to the same status as the diatonic and enharmonic genera.

The Greater and Lesser Perfect Systems

However the early evolution of the Greek musical system actually occurred, the result came to be schematized as the Perfect Immutable System. Its construction was as follows: two identical tetrachords of any genus and a disjunctive tone (9/8) formed a central heptatonic scale which became the core of the system. Another identical tetrachord was then added by conjunction at both ends of the scale and disjunctive tone was patched on at the bottom of the whole array. A fifth tetrachord, synemmenon, was inserted conjunctly into the middle of the system to recall the ancient heptachord and to facilitate commonly occurring modulations at the fourth. This supernumerary tetrachord was also a useful pedagogical device to illustrate unusual intervals (Erickson 1965; Steinmayer 1985).

The final results consisted of sets of five tetrachords linked by conjunction and disjunction into arrays of fifteen notes spanning two octaves. These systems, in turn, could be transposed into numerous pitch keys or tonoi, at intervals roughly a semitone apart according to the later authors.

The subset of four alternately conjunct and disjunct tetrachords (hypaton, meson, diezeugmenon, and hyperbolaion) was termed the greater perfect (or complete) system (σψστεμα τελειον μειζον). The three conjunct tetrachords (hypaton, meson, and synemmenon), was called the Lesser Perfect (or Complete) System (σψστημα τελειον ελαττον or ελασσον). Their union was called variously the Changeless System or the Perfect Immutable System (συστημα τελειον αμεταβολον) by different authors.

The Perfect Immutable System

By the fourth century BCE, the Greek theorists had analyzed the scales or harmoniai of their music into sections of this theoretical two octave gamut. This 15-note span is conventionally transcribed into our notation as lying between A and a'. The Perfect Immutable System could be tuned to each of the three genera, and while in theory all five of the tetrachords must be the same, in practice mixed tetrachords and considerable chromaticism occurred. Not only was the diatonic lichanos meson (D in the Dorian or E mode) added, but other extrascalar notes led to successions of more than two semitones (Winnington-Ingram 1936).

6-5 depicts the Perfect Immutable System in its theoretical form and in its two most historically important intonations.

The fixed notes (hestotes) of the Perfect Immutable System were proslambanomenos, hypate hypaton, hypate meson, mese, paramese, nete diezeugmenon, nete hyperbolaion, and nete synemmon. The moveable tones (κινουμενοι) were the parhypatai, the lichanoi, the tritai, and the paranetai of each genus.

Lichanos hypaton, also called hyperhypate, a diatonic note a whole tone (9/8 in Archytas's and most other just tunings) below the tonic, was added to the Dorian octave species in the chromatic and enharmonic genera in the harmoniai of Aristides Quintilianus, certain planetary scales, and the Euripides fragment (ibid.).

Erickson (1965) and Vogel (1963, 1975) have shown that a number of interesting tetrachords occur in the region where the synemmenon tetrachord overlaps with the diezeugmenon tetrachord in Archytas's system. These include the later and historically important 16/15 · 9/8 · 10/9 (Ptolemy's syntonic diatonic), 16/15 · 10/9 · 9/8 (Didymos's diatonic), the three permutations of the Pythagorean diatonic, 256/243 · 9/8 · 9/8, (90 + 204 + 204 cents), the Pythagorean chromatic 32/27 · 2187/2048 · 256/243 (294 +

114 + 90 cents), and Avicenna's chromatic 7/6 · 36/35 · 10/9 (267 + 49 + 182 cents). Some unusual divisions such as 28/27 · 81/70 · 10/9 (63 + 253 + 182 cents), 28/27 · 2187/1792 · 256/243 (63 + 345 + 90 cents), 16/15 · 35/32 · 8/7 (112 + 155 + 231 cents), 16/15 · 1215/1024 · 256/243 (112 + 296 + 90 cents), 7/6 · 81/80 · 9/8 (267 + 22 + 204 cents), 32/27 · 81/80 · 10/9 (294 + 22 + 182 cents), 28/27 · 64/63 · 81/64 (63 + 22 + 408 cents), 6/5 · 135/128 · 256/243 (316 + 92 + 90 cents), and 256/243 · 81/80 · 5/4 (90 + 22 + 386 cents) are also found here. Notable are the intervals of 253 cents, another possible tuning for the ekbole, the neutral third of 345 cents, the three-quarter tone 35/32 (155 cents), and the minor whole tone 10/9.

The alternate tunings 16/15 and 28/27 for the first interval of the synemmenon tetrachord may have been used in order to obtain the spondeiasmos, an interval of three dieses approximating 150 cents, mentioned by Bacchios (Steinmayer 1985; Winnington-Ingram 1932). These intervals would measure 35/32 (155 cents) as the difference between 14/9 and 64/45, or 243/224 (141 cents) as the difference between 112/81 and 3/2. The in-

6-5. *The Perfect Immutable System in the diatonic, chromatic, and enharmonic genera, tuned according to Archytas's and Pythagorean tuning. The transcription is in the natural key to avoid accidentals and the mistaken late shift of emphasis from Dorian to Hypolydian (Henderson 1957). The - and 𝄫 indicate that these are different pitches in the enharmonic genus. Erickson (1965) proposes 64/45 as an alternative tuning for trite synemmenon.*

	TRANSCRIPTION			ARCHYTAS			PYTHAGOREAN		
	DIA.	CHR.	ENH.	DIA.	CHR.	ENH.	DIA.	CHR.	ENH.
PROSLAMBANOMENOS	A	A	A	2/3	2/3	2/3	2/3	2/3	2/3
HYPATE HYPATON	B	B	B	3/4	3/4	3/4	3/4	3/4	3/4
PARHYPATE HYPATON	C	C	C-	7/9	7/9	7/9	64/81	64/81	384/499
LICHANOS HYPATON	D	D♭	D𝄫	8/9	27/32	4/5	8/9	27/32	64/81
HYPATE MESON	E	E	E	1/1	1/1	1/1	1/1	1/1	1/1
PARHYPATE MESON	F	F	F-	28/27	28/27	28/27	256/243	256/243	512/499
LICHANOS MESON	G	G♭	G𝄫	32/27	9/8	16/15	32/27	9/8	256/243
MESE	a	a	a	4/3	4/3	4/3	4/3	4/3	4/3
PARAMESE	b	b	b	3/2	3/2	3/2	3/2	3/2	3/2
TRITE DIEZEUGMENON	c	c	c-	14/9	14/9	14/9	128/81	128/81	768/499
PARANETE DIEZEUGMENON	d	d♭	d𝄫	16/9	27/16	8/5	16/9	27/16	128/81
NETE DIEZEUGMENON	e	e	e	2/1	2/1	2/1	2/1	2/1	2/1
TRITE HYPERBOLAION	f	f	f-	56/27	56/27	56/27	512/243	512/243	1024/499
PARANETE HYPERBOLAION	g	g♭	g𝄫	64/27	9/4	32/15	64/27	9/4	512/243
NETE HYPERBOLAION	a'	a'	a'	8/3	8/3	8/3	8/3	8/3	8/3
TRITE SYNEMMENON (28/27)	b♭	b♭	b♭-	112/81	112/81	112/81	1024/729	1024/729	2048/1497
PARANETE SYNEMMENON	c	c♭	c𝄫	128/81	3/2	64/45	128/81	3/2	1024/729
NETE SYNEMMENON	d	d	D	16/9	16/9	16/9	16/9	16/9	16/9

3/2. The interval of three dieses also appears in Archytas's chromatic as the difference between the 28/27 and the 9/8. In many cases the scales containing these tetrachords would be mixed, but deliberately mixed scales were not unknown. 6-6 lists some varieties of mixed scales recorded by Ptolemy in the second century CE.

The scales actually employed in Greek music are a matter of some confusion because of the paucity of extant musical examples and the variety of theoretical works from different traditions written over a period of several centuries (fourth century BCE to fourth century CE). In the theoretical treatises, the seven octave species or circular permutations of the basic heptatonic scale are singled out and given names derived from early tribal groups. These scales are notated in all three genera in 6-7. Their intervals and notes are in shown in ratios for both Archytas's and Pythagorean tuning in 6-8 and 6-9. 6-10 gives the diatonic form in Ptolemy's syntonic diatonic (16/15 · 9/8 · 10/9), and 6-11 gives the retrograde of this genus (10/9 · 9/8 · 16/15). The Lydian mode in the former tuning is the standard just intonation of the major scale, and the latter is that of the natural minor mode (see chapter 7).

For the Pythagorean tuning of the enharmonic, I have used Boethius's much later arithmetic division of the pyknon, as the actual tuning prior to Archytas is not known. Since the division of the semitone in both tetra-

6-6. *Scales in common use according to Ptolemy. In the text, the names of the tunings are always given in plural form. (1), not the ditonic or Pythagorean, appears to have been the standard diatonic. On the kithara, in the Hypodorian mode it was called tritai; in the Phrygian, hypertropa. (2a) is given in two forms in different places in the* Harmonics; *the intense chromatic (1:84), where it is mistranslated as "diatonic chromatic," and the soft chromatic (2:208). The tables (2:178) use the intense chromatic; the soft chromatic fits the sense of the name better. On the kithara, (2b) in the Hypodorian mode is called tropoi or tropikoi. In the Dorian mode on the kithara, (3) is called parypatai. (4) is in the Hypophrygian mode. (5), in the Dorian mode, is given variously as either pure tonic diatonic or a mixture of tonic diatonic and intense and is also referred to as metabolika. (6) is from Avicenna (D'Erlanger 1935, 2:239), who sometimes approximated complex ratios like 72/65 with superparticulars of similar magnitude such as 22/21, but the exact ratio is clear from the context.*

1. STEREA, A LYRA TUNING: TONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 14/9 16/9 2/1

2. MALAKA, A LYRA TUNING: SOFT OR INTENSE CHROMATIC AND TONIC DIATONIC
A. 1/1 28/27 10/9 4/3 3/2 14/9 16/9 2/1
B. 1/1 22/21 8/7 4/3 3/2 14/9 16/9 2/1

3. METABOLIKA, ANOTHER LYRA TUNING: SOFT DIATONIC AND TONIC DIATONIC
1/1 21/20 7/6 4/3 3/2 14/9 16/9 2/1

4. IASTI-AIOLIKA, A KITHARA TUNING: TONIC DIATONIC AND DITONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 27/16 16/9 2/1

5. IASTIA OR LYDIA, KITHARA TUNINGS: INTENSE DIATONIC AND TONIC DIATONIC
1/1 28/27 32/27 4/3 3/2 8/5 9/5 2/1

6. A MEDIEVAL ISLAMIC SCALE OF ZALZAL FOR COMPARISON
1/1 9/8 81/64 4/3 40/27 130/81 16/9 2/1

chords was completed only near end of the fourth century BCE, the division may not have been standardized and was most likely done by ear during the course of the melody (Winnington-Ingram 1928), in which case the approximate equality of the dieses in Boethius's tuning probably captures the flavor of the scale adequately. Euler's eighteenth-century tuning (Euler [1739] 1960, and Catalog number 79) is similar and considerably simpler. An impractical, if purely Pythagorean, solution (number 81) as well as some other approximations are given in the Main Catalog.

Although these scales are analogous to the "white key" modes, the latter are named out of order due to a misunderstanding in early medieval times.

TONIC	NAME	MESE
	Diatonic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D)
B	MIXOLYDIAN, HYPERDORIAN	E
C	LYDIAN	F
D	PHRYGIAN	G
E	DORIAN	a
F	HYPOLYDIAN	b
G	HYPOPHRYGIAN, IONIAN	c
a	HYPODORIAN, AEOLIAN	d
	Chromatic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D♭)
B	MIXOLYDIAN, HYPERDORIAN	E
C	LYDIAN	F
D♭	PHRYGIAN	G♭
E	DORIAN	a
F	HYPOLYDIAN	b
G♭	HYPOPHRYGIAN, IONIAN	c
a	HYPODORIAN, AEOLIAN	DB
	Enharmonic	
(A	HYPERMIXOLYDIAN, HYPERPHRYGIAN, LOCRIAN	D𝄫)
B	MIXOLYDIAN, HYPERDORIAN	E
C-	LYDIAN	F-
D𝄫	PHRYGIAN	G𝄫
E	DORIAN	a
F-	HYPOLYDIAN	b
G𝄫	HYPOPHRYGIAN, IONIAN	c-
a	HYPODORIAN, AEOLIAN	d𝄫

6-7. *The octave species in all three genera. The traditional names are given first and alternate ones subsequently. The Hypermixolydian was denounced by Ptolemy as otiose and by the city of Argos as illegal (Winnington-Ingram 1936). This transcription uses the natural key for clarity. Late theorists mistakenly built the system and notation about the F mode (Hypolydian) rather than the correct E mode (Dorian) (Henderson 1957). Although the Dorian, Phrygian, and Lydian modes have distinctive tetrachordal forms, these forms were never named after their parent modes by any of the Greek theorists. In the chromatic and enharmonic genera the tonics of the species are transformed. An alternative nomenclature for the enharmonic tetrachord is E E+ F A. The mese kata thesin is four scale degrees above the tonic with which it usually makes an interval of a perfect fourth.*

Diatonic (28/27 · 8/7 · 9/8)

MIXOLYDIAN (B – b)

1/1 28/27 32/27 4/3 112/81 128/81 16/9 2/1
28/27 · 8/7 · 9/8 · 28/27 · 8/7 · 9/8 · 9/8

LYDIAN (C – c)

1/1 8/7 9/7 4/3 32/21 12/7 27/14 2/1
8/7 · 9/8 · 28/27 · 8/7 · 9/8 · 9/8 · 28/27

PHRYGIAN (D – d)

1/1 9/8 7/6 4/3 3/2 27/16 7/4 2/1
9/8 · 28/27 · 8/7 · 9/8 · 9/8 · 28/27 · 8/7

DORIAN (E – e)

1/1 28/27 32/27 4/3 3/2 4/9 16/9 2/1
28/27 · 8/7 · 9/8 · 9/8 · 28/27 · 8/7 · 9/8

HYPOLYDIAN (F – f)

1/1 8/7 9/7 81/56 3/2 12/7 27/14 2/1
8/7 · 9/8 · 9/8 · 28/27 · 8/7 · 9/8 · 28/27

HYPOPHRYGIAN (G – g)

1/1 9/8 81/64 21/16 3/2 27/16 7/4 2/1
9/8 · 9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 8/7

HYPODORIAN (A – a)

1/1 9/8 7/6 4/3 3/2 14/9 16/9 2/1
9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 8/7 · 9/8

Chromatic (28/27 · 243/224 · 32/27)

MIXOLYDIAN (B – b)

1/1 28/27 9/8 4/3 112/81 3/2 16/9 2/1
28/27 · 243/224 · 32/27 · 28/27 · 243/224 · 32/27 · 9/8

LYDIAN (C – c)

1/1 243/224 9/7 4/3 81/56 12/7 27/14 2/1
243/224 · 32/27 · 28/27 · 243/224 · 32/27 · 9/8 · 28/27

PHRYGIAN (D♭ – d♭)

1/1 32/27 896/729 4/3 128/81 16/9 448/243 2/1
32/27 · 28/27 · 243/224 · 32/27 · 9/8 · 28/27 · 243/224

DORIAN (E – e)

1/1 28/27 9/8 4/3 3/2 14/9 27/16 2/1
28/27 · 243/224 · 32/27 · 9/8 · 28/27 · 243/224 · 32/27

HYPOLYDIAN (F – f)

1/1 243/224 9/7 81/56 3/2 729/448 27/14 2/1
243/224 · 32/27 · 9/8 · 28/27 · 243/224 · 32/27 · 28/27

HYPOPHRYGIAN (G♭ – g♭)

1/1 32/27 4/3 112/81 3/2 16/9 448/243 2/1
32/27 · 9/8 · 28/27 · 243/224 · 32/27 · 28/27 · 243/224

HYPODORIAN

1/1 9/8 7/6 81/64 3/2 14/9 27/16 2/1
9/8 · 28/27 · 243/224 · 32/27 · 28/27 · 243/224 · 32/27

Enharmonic (28/27 · 36/35 · 5/4)

MIXOLYDIAN (B – b)

1/1 28/27 16/15 4/3 112/81 64/45 16/9 2/1
28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4 · 9/8

LYDIAN (C- – c-)

1/1 36/35 9/7 4/3 48/35 12/7 27/14 2/1
36/35 · 5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27

PHRYGIAN (D𝄫 – d𝄫)

1/1 5/4 35/27 4/3 5/3 15/8 35/18 2/1
5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35

DORIAN (E – e)

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

HYPOLYDIAN (F- – f-)

1/1 36/35 9/7 81/56 3/2 54/35 27/14 2/1
36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4 · 28/27

HYPOPHRYGIAN (G𝄫 – g𝄫)

1/1 5/4 45/32 35/24 3/2 15/8 35/18 2/1
5/4 · 9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35

HYPODORIAN (A – a)

1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1
9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4

6-8. *The intervals of the octave species in all three genera in Archytas's tuning.*

Diatonic (256/243 · 9/8 · 9/8)

MIXOLYDIAN (B – b)

1/1 256/243 32/27 4/3 1024/729 128/81 16/9 2/1

256/243 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 9/8

LYDIAN (C – c)

1/1 9/8 81/64 4/3 3/2 27/16 243/128 2/1

9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 9/8 · 256/243

PHRYGIAN (D – d)

1/1 9/8 32/27 4/3 3/2 27/16 16/9 2/1

9/8 · 256/243 · 9/8 · 9/8 · 9/8 · 256/243 · 9/8

DORIAN (E – e)

1/1 256/243 32/27 4/3 3/2 128/81 16/9 2/1

256/243 · 9/8 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8

HYPOLYDIAN (F – f)

1/1 9/8 81/64 729/512 3/2 27/16 243/128 2/1

9/8 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 256/243

HYPOPHRYGIAN (G – g)

1/1 9/8 81/64 4/3 3/2 27/16 16/9 2/1

9/8 · 9/8 · 256/243 · 9/8 · 9/8 · 256/243 · 9/8

HYPODORIAN (A – a)

1/1 9/8 32/27 4/3 3/2 128/81 16/9 2/1

9/8 · 256/243 · 9/8 · 9/8 · 256/243 · 9/8 · 9/8

Chromatic (256 · 2187/2028 · 32/27)

MIXOLYDIAN (B – b)

1/1 256/243 9/8 4/3 1024/729 3/2 16/9 2/1

256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27 · 9/8

LYDIAN (C – c)

1/1 2187/2048 81/64 4/3 729/512 27/16 243/128 2/1

2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27 · 9/8 · 256/243

PHRYGIAN (D♭ – d♭)

1/1 32/27 8192/6561 4/3 128/81 16/9 4096/2187 2/1

32/27 · 256/243 · 2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048

DORIAN (E – e)

1/1 256/243 9/8 4/3 3/2 128/81 27/16 2/1

256/243 · 2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048 · 32/27

HYPOLYDIAN (F – f)

1/1 2187/2048 81/64 729/512 3/2 6561/4096 243/128 2/1

2187/2048 · 32/27 · 9/8 · 256/243 · 2187/2048 · 32/27 · 256/243

HYPOPHRYGIAN (G♭ – g♭)

1/1 32/27 4/3 729/512 3/2 16/9 4096/2187 2/1

32/27 · 9/8 · 256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048

HYPODORIAN (A – a)

1/1 9/8 32/27 81/64 3/2 128/81 27/16 2/1

9/8 · 256/243 · 2187/2048 · 32/27 · 256/243 · 2187/2048 · 32/27

Enharmonic (512/499 · 499/486 · 81/64)

MIXOLYDIAN (B – b)

1/1 512/499 256/243 4/3 2048/1497 1024/729 16/9 2/1

512/499 · 499/486 · 81/64 · 512/499 · 499/486 · 81/64 · 9/8

LYDIAN (C-- c-)

1/1 499/486 499/384 4/3 998/729 499/288 499/256 2/1

499/486 · 81/64 · 512/499 · 499/486 · 81/64 · 9/8 · 512/499

PHRYGIAN (D𝄫 – d𝄫)

1/1 81/64 648/499 4/3 27/16 243/128 972/499 2/1

81/64 · 512/499 · 499/486 · 81/64 · 9/8 · 512/499 · 499/486

DORIAN (E – e)

1/1 512/499 256/243 4/3 3/2 768/499 128/81 2/1

512/499 · 499/486 · 81/64 · 9/8 · 512/499 · 499/486 · 81/64

HYPOLYDIAN (F-- f-)

1/1 499/486 499/384 1497/1024 3/2 499/324 499/256 2/1

499/486 · 81/64 · 9/8 · 512/499 · 499/486 · 81/64 · 512/499

HYPOPHRYGIAN (G𝄫 – g𝄫)

1/1 81/64 729/512 729/499 3/2 243/128 972/499 2/1

81/64 · 9/8 · 512/499 · 499/486 · 81/64 · 512/499 · 499/486

HYPODORIAN (A – a)

1/1 9/8 576/499 32/27 3/2 768/499 128/81 2/1

9/8 · 512/499 · 499/486 · 81/64 · 512/499 · 499/486 · 81/64

6-9. *The intervals of the octave species in Pythagorean tuning. The tuning of the pre-Archytas enharmonic is not known, but at first it had undivided semitones, obtaining the pyknon later. Boethius's tuning is used here.*

Although they are conventionally presented as sections of the two octave gamut, they were actually retunings of the central octave so that the sequences of intervals corresponding to the cyclic modes fell on the notes of the Perfect Immutable System (hypate meson to nete diezeugemenon, e to e'). These abstract sequences of intervals are shown in 6-12. Thus, in the Dorian tonos, the interval sequence of the Dorian mode filled the central octave; in the Phrygian, the Phrygian sequence was central and the Dorian, a tone higher. In the Hypolydian tonos, the initial A, proslambanomenos, was raised a semitone, as was its octave, mese, the supposed tonal center of the whole system.

From the original set of seven pitch keys (tonoi), a later set of thirteen or fifteen theoretical keys at more or less arbitrary semitonal intervals developed, irrespective of genus (Crocker 1966; Winnington-Ingram 1936). In Roman times, the theorists moved the entire system up a semitone so

6-10. *The intervals of the octave species of Ptolemy's intense diatonic genus. See figures 6-3 and 6-6 for names of notes. The diatonic tetrachord is 16/15 · 9/8 · 10/9. The Lydian mode in this tuning is the major mode in just intonation. The Hypodorian or A mode is not the minor mode as the fourth degree is 27/20 instead of 4/3.*

MIXOLYDIAN (B – b)
1/1 16/15 6/5 4/3 64/45 8/5 16/9 2/1
16/15 · 9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 9/8

LYDIAN (C – c)
1/1 9/8 5/4 4/3 3/2 5/3 15/8 2/1
9/8 · 10/9 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15

PHRYGIAN (D – d)
1/1 10/9 32/27 4/3 40/27 5/3 16/9 2/1
10/9 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 9/8

DORIAN (E – e)
1/1 16/15 6/5 4/3 3/2 8/5 9/5 2/1
16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

HYPOLYDIAN (F – f)
1/1 9/8 5/4 45/32 3/2 27/16 15/8 2/1
9/8 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 16/15

HYPOPHRYGIAN (G – g)
1/1 10/9 5/4 4/3 3/2 5/3 16/9 2/1
10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 16/15 · 9/8

HYPODORIAN (A – a)
1/1 9/8 6/5 27/20 3/2 8/5 9/5 2/1
9/8 · 16/15 · 9/8 · 10/9 · 16/15 · 9/8 · 10/9

6-11. *The intervals of the octave species of the Ptolemy's intense diatonic genus, reversed. The diatonic tetrachord is 10/9 · 9/8 · 16/15. The Lydian or C mode in this tuning is the minor mode in just intonation. The Dorian or E mode is not the major mode as the second degree is 10/9 instead of 9/8. This scale transposed to C is John Redfield's tuning for the major scale (Redfield 1928).*

MIXOLYDIAN (B – b)
1/1 10/9 5/4 4/3 40/27 5/3 16/9 2/1
10/9 · 9/8 · 16/15 · 10/9 · 9/8 · 16/15 · 9/8

LYDIAN (C – c)
1/1 9/8 6/5 4/3 3/2 8/5 9/5 2/1
9/8 · 16/15 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9

PHRYGIAN (D – d)
1/1 16/15 32/27 4/3 64/45 8/5 16/9 2/1
16/15 · 10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 9/8

DORIAN (E – e)
1/1 10/9 5/4 4/3 3/2 5/3 15/8 2/1
10/9 · 9/8 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15

HYPOLYDIAN (F – f)
1/1 9/8 6/5 27/20 3/2 27/16 9/5 2/1
9/8 · 16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 10/9

HYPOPHRYGIAN (G – g)
1/1 16/15 6/5 4/3 3/2 8/5 16/9 2/1
16/15 · 9/8 · 10/9 · 9/8 · 16/15 · 10/9 · 9/8

HYPODORIAN (A – a)
1/1 9/8 5/4 45/32 3/2 5/3 15/8 2/1
9/8 · 10/9 · 9/8 · 16/15 · 10/9 · 9/8 · 16/15

6-12. *Interval sequences of the octave species of the abstract tetrachord* a · b · c. a · b · c = *4/3* (c = *4/3*ab) *in just intonation or* a + b + *500* - a - b *with the disjunctive tone equaling 200 cents in the zero modulo 12 equal temperaments. In the Main Catalog,* c *is equal to the CI.*

MIXOLYDIAN *a · b · c · a · b · c ·* 9/8	HYPOLYDIAN *b · c ·* 9/8 *· a · b · c · a*
LYDIAN *b · c · a · b · c ·* 9/8 *· a*	HYPOPHRYGIAN *c ·* 9/8 *· a · b · c · a · b*
PHRYGIAN *c · a · b · c ·* 9/8 *· a · b*	HYPODORIAN 9/8 *· a · b · c · a · b · c*
DORIAN *a · b · c ·* 9/8 *· a · b · c*	

that the central octave began on either E or F in modern notation. In this final form, however, the central octave had the interval sequence of the Hypolydian mode rather than the Dorian.

The modal retunings could also be considered as transpositions of the entire Perfect Immutable System. The order of the keys ran in the opposite direction to that of the homonymous octave species and the octave species could be described either by the positions of their interval sequences in relation to the untransposed Dorian or by the relative pitch of the entire Perfect Immutable System. This duality is reflected in the two nomenclatures employed by Ptolemy, the "onomasia kata thesin" (by position) and "onomasia kata dynamin" (by function). The thetic nomenclature in the natural key is used in the tables of this chapter and chapter 8 as it is the same for all tonoi. The dynamic refers all notes to the Dorian tonos for which the thetic and dynamic nomenclatures are identical.

6-13. *Vogel's transcription of the Greek notations. Only the upper octave from mese to nete hyperbolaion is shown. Vogel's German notation has been transcribed into the American form. His notes have been transposed up an octave, and those marked with a bar in the original are given a + here. 512/405 (406 cents) replaces 81/64 (408 cents), in Vogel's tuning. In the upper half of the scale, 2048/1215 replaces 27/16.*

NOTE	RATIO	NOTATION
MESE	1/1	A
TRITE SYNEMMENON	28/27	B♭-
PARANETE SYNEMMENON	16/15 (ENHARMONIC)	B♭+
PARANETE SYNEMMENON, PARAMESE	9/8 (CHROMATIC)	B
TRITE DIEZEUGMENON	7/6	C-
PARANETE SYNEMMENON	32/27 (DIATONIC)	C
PARANETE DIEZEUGMENON	6/5 (ENHARMONIC)	C+
	896/729	D♭-
	512/405 (CHROMATIC)	D♭+
	4/3 (DIATONIC)	D
NETE SYNEMMENON	112/81	E♭-
	64/45	E♭
NETE DIEZEUGMENON	3/2	E
TRITE HYPERBOLAION	14/9	F-
PARANETE HYPERBOLAION	8/5 (ENHARMONIC)	F+
	128/81	F
	3584/2187	G♭-
PARANETE HYPERBOLAION	2048/1215 (CHROMATIC)	G♭
	16/9 (DIATONIC)	G
	448/243	A♭-
	256/135	A♭
NETE HYPERBOLAION	2/1	A

6-14. *Unusual tetrachords in Vogel's transcription.*

RATIOS	CENTS
64/63 · 81/80 · 35/27	27 + 22 + 449
81/80 · 2240/2187 · 9/7	22 + 41 + 435
36/35 · 2240/2187 · 81/84	49 + 41 + 408
36/35 · 256/243 · 315/256	49 + 90 + 359
64/63 · 16/15 · 315/256	27 + 112 + 359
64/63 · 2187/2048 · 896/729	27 + 114 + 357
896/729 · 36/35 · 135/128	357 + 49 + 92
28/27 · 256/243 · 2187/1792	63 + 90 + 345
16/15 · 2240/2187 · 2187/1792	112 + 41 + 345
28/27 · 128/105 · 135/128	63 + 343 + 92
6/5 · 35/32 · 64/63	316 + 155 + 27
6/5 · 2240/2187 · 243/224	316 + 41 + 141
7168/6561 · 36/35 · 1215/1024	153 + 49 + 296
16/15 · 1215/1024 · 256/243	112 + 296 + 90
28/27 · 1024/945 · 1215/1024	63 + 139 + 296
7/6 · 1024/945 · 135/128	267 + 139 + 92
28/27 · 81/70 · 10/9	63 + 253 + 182
81/70 · 2240/2187 · 9/8	253 + 41 + 204
81/70 · 256/243 · 35/32	253 + 90 + 155
135/128 · 7168/6561 · 81/70	92 + 153 + 253
16/15 · 280/243 · 243/224	112 + 245 + 141
36/35 · 9/8 · 280/243	49 + 204 + 245
8/7 · 81/80 · 280/243	231 + 22 + 245
9/8 · 7168/6561 · 243/224	204 + 153 + 141
9/8 · 4096/3645 · 135/128	204 + 202 + 92
35/32 · 1024/945 · 9/8	155 + 139 + 204
4096/3645 · 35/32 · 243/224	202 + 155 + 141

The Greeks named the modes from their keynotes as octave species of the Perfect Immutable System, while the medieval theorists named them in order of their transpositions (Sachs 1943). The two concepts became confused by the time of Boethius. For this reason the names of the ecclesiastical modes are different from those of ancient Greece. In more recent periods, other ecclesiastical nomenclatures were developed.

Greek alphabetic notations

In addition to the thetic and dynamic nomenclatures, which were really tablatures derived from the names of the strings of the kithara or similar instrument, there were two alphabetical cipher notations, the vocal and the instrumental. These were recorded for the each of the tonoi in all three genera by the theorist Alypius. The independent elucidation of Alypius's tables by Bellermann (1847) and Fortlage (1847) have permitted scholars to transcribe the few extant fragments of Greek music into modern notation.

Vogel (1963, 1967) has translated these cipher notations into a tuning system based on Archytas's and Pythagoras's genera (6-4). This set of tones includes a number of unusual tetrachords, most of which occur in several permutations (6-13). Some of these are good approximations to the neo-Aristoxenian types: 50 + 100 + 350 cents, 50 + 150 + 300 cents, 50 + 250 + 200 cents, and 150 + 150 + 200 cents of chapter 4.

The Greek notations, however, were not entirely without ambiguity, and some uncertainly exists over the meaning of certain presumed "enharmonic" equivalences, i.e. two notes of the same pitch written differently. Kathleen Schlesinger developed her somewhat fantastic theories, detailed in chapter 8, in part from deliberations on the apparent anomalies of these notations.

Concise descriptions of the notational systems may be found in Sachs (1943) and Henderson (1957).

The oldest harmoniai or modes

Although the melodic canons laid down by Aristoxenos (330 BCE) stated that the smallest interval the melody could move from the pyknon was a whole tone and that notes four or five positions apart must make either perfect fourths or fifths, both literary evidence and the surviving fragments attest to mixed scales and chromaticism (Winnington-Ingram 1936), as mentioned previously. A late writer, Aristides Quintilianus, gave a list of what he said were the scales approved by Plato in the *Republic*. These scales are in the enharmonic genus and depart quite strongly from the conventional octave species of 6-7. Since it is known that both diatonic and chromatic scales of the same name existed, it is tempting to try to reconstruct them. 6-4 contains Aristides's enharmonic harmoniai, Henderson's (1942) diatonic versions, and my own chromatic and diatonic forms. The chromatic versions are based on Winnington-Ingram's indication that there is literary evidence for certain chromatic versions (1936). The diatonic harmoniai are from Henderson (1942), except in the cases of the Syntonolydian and Iastian where I have supplied a second diatonic which I feel better preserves the melodic contours. In the enharmonic and chromatic forms of some of the harmoniai, it has been necessary to use both a d and either a d𝄳 or d𝄫 because of the non-heptatonic nature of these scales. C and F are synonyms for d𝄫 and g𝄫. The appropriate tunings for these scales are those of Archytas (Mountford 1923) and Pythagoras.

These scales are very important evidence for the use of extrascalar tones (diatonic lichanos meson, called hyperhypate) and scalar gaps, which were alluded to by Aristoxenos as an indispensable ingredient in determining the ethos of the mode. Furthermore, one of the fragments, a portion of the first stationary chorus of Euripides's *Orestes*, uses hyperhypate and the enharmonic in such a way as to prove that the middle tone of the pyknon (mesopyknon) was not merely a grace note, but a full member of the scale (Winnington-Ingram 1936).

Ptolemy's mixed scales

Still more remote from the conventional theory are the mixed scales listed by Ptolemy in the *Harmonics*. These scales are ones that he said were in common use by players of the lyra and kithara in Alexandria in the second century CE (6-6). These scales bear some resemblance to modern Islamic modes containing 3/4-tone intervals, as does Ptolemy's equable diatonic, 12/11 · 11/10 · 10/9. They offer important support and evidence for the combination of tetrachords of varying genera and species to generate new musical materials.

Permutation of intervals

Although traditional techniques can generate a wealth of interesting material for musical exploration, the Greek writers suggested only a small fraction of the possibilities inherent in the permutations and combinations of tetrachords. While Aristoxenos mentioned the varying arrangements of the intervals of the tetrachord in the different octave species, the Islamic theorists, such as Safiyu-d-Din, gave lengthy tables of all the permutational forms of tetrachords with two and three different intervals. However, the construction of 5-, 6-, and 7-tone scales from permuted tetrachords and trichords (gapped tetrachords) has been studied most thoroughly by the composer Lou Harrison (1975). Harrison constructed scales from all the permutations of the tetrachords and trichords and allowed different permutations in the upper and lower parts of the scale.

In chapter 5, the melodic properties of scales constructed of either identical or dissimilar tetrachords, irrespective of permutational order, are analyzed according to the perception theories of David Rothenberg (1969, 1975, 1978; also Chalmers 1975).

Wilson's permutations and modulations

Perhaps the most sophisticated use to date of tetrachordal interval permutation in a generative sense is Ervin Wilson's derivation of certain North Indian thats (raga-scales) and their analogs (Wilson 1986a; 1987). In "The Marwa Permutations" (1986a), Wilson's procedure is to permute the order of the sequential fourths of heptatonic scales constructed from two identical tetrachords. These sequential fourths are computed in the usual manner by starting with the lowest note of one of the modes and counting three melodic steps upwards. The process is continued until the cycle is complete and one is back to the original tone. The resulting seven fourths are the same as the adjacent fourths of the difference matrices of chapter 5, but in a different order. In abstract terms, if the intervals of the tetrachord are *a* · *b/a* · 4/3*b*, the scale is 1/1 *a b* 4/3 3/2 3*a*/2 3*b*/2, and 2/1. The sequential fourths from 1/1 are thus 4/3, 3/2*a*, 3*a*/2*b*, 9*b*/8, 4/3, 4/3, and 4/3. It is clear that these fourths must be of at least two different sizes even in Pythagorean intonation.

While holding the position of one fourth constant to avoid generating cyclic permutations or modes, pairs of fourths are exchanged to create new sequences of intervals in general not obtainable by the traditional modal operations. Both the choice of the positionally fixed fourth and the arrangement of the tetrachordal intervals affect the spectrum of scales obtainable from a given genus.

6-15 illustrates this process with the enharmonic genus of Archytas. The exchange of the second and third fourths converts the upper tetrachord into

6-15. *Permutations of sequential fourths. See Wilson 1986 for further details. This example begins with the Dorian mode of the standard ascending form for clarity and consistency with other sections of this treatise. The sizes of the fourths range from 6/5 (316 cents) to 35/24 (653 cents). Interval 7 in the original sequence is a fixed fourth. The pair of permuted fourths are in boldface. The last tetrachord is Archytas's diatonic.*

ORIGINAL SCALE

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

	FOURTHS	SIZE
1.	1/1 *to* 4/3	4/3
2.	4/3 *to* 8/5	6/5
3.	8/5 *to* 16/15	4/3
4.	16/15 *to* 14/9	35/24
5.	14/9 *to* 28/27	4/3
6.	28/27 *to* 3/2	81/56
7.	3/2 *to* 2/1	4/3

ORIGINAL SEQUENCE

1	2	3	4	5	6	7
4/3	**6/5**	**4/3**	35/24	4/3	81/56	(4/3)

PERMUTED SEQUENCE

1	3	2	4	5	6	7
4/3	**4/3**	**6/5**	35/24	4/3	81/56	(4/3)

NEW SCALE

1/1 28/27 16/15 4/3 3/2 14/9 16/9 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 8/7 · 9/8

Archytas's diatonic and yields a mixed scale, half enharmonic and half diatonic. Further application of this principle produces additional scales until the original sequence is restored. Each of these scales could be modally (cyclically) permuted as well.

Wilson derives a number of the thats of North Indian ragas by operating on various arrangements of the tetrachords 256/243 · 9/8 · 9/8, 16/15 · 9/8 · 10/9, 28/27 · 8/7 · 9/8, 16/15 · 135/128 · 32/27, and 10/9 · 10/9 · 27/25. He then generates analogs of these scales from other tetrachords, including those with undecimal intervals.

In his 1987 paper, Wilson described a complementary technique of modulation ("The Purvi Modulations"). This technique makes use of the fact that at least one of the fourths differs greatly in size from the rest. The exceptional fourth may be abstracted from the linear fourth sequence and interpolated between successive pairs to generate derived scales. At the end of seven such interpolations, the linear sequence is cyclically permuted by one position and the process of interpolation continued. After 42 steps the

6-16. *Modulations by sequential fourths. This example begins with the Dorian mode for consistency with other sections of this treatise. The sizes of the fourths range from 6/5 (316 cents) to 35/24 (653 cents). In the original sequence the exceptional fourth is in bold face. In the rotated sequence the scale has been modally permuted to separate the exceptional fourth (in boldface) from the rest. In the first modulated sequence the 6/5 (in boldface) has been interpolated between fourths 7 and 1 of the original series. In the second modulated sequence the 6/5 (in boldface) has been interpolated between fourths 3 and 4 of the original series. The new tetrachord is Archytas's diatonic.*

ORIGINAL SCALE

1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35 · 5/4

	FOURTHS	SIZE
1.	1/1 TO 4/3	4/3
2.	4/3 TO 8/5	6/5
3.	8/5 TO 16/15	4/3
4.	16/15 TO 14/9	35/24
5.	14/9 TO 28/27	4/3
6.	28/27 TO 3/2	81/56
7.	3/2 TO 2/1	4/3

ORIGINAL SEQUENCE

1	2	3	4	5	6	7
4/3	**6/5**	4/3	35/24	4/3	81/56	4/3

ROTATED SEQUENCE

3	4	5	6	7	1	2
4/3	35/24	4/3	81/56	4/3	4/3	**6/5**

NEW SCALE

1/1 5/4 35/27 4/3 5/3 15/8 35/18 2/1
5/4 · 28/27 · 36/35 · 5/4 · 9/8 · 28/27 · 36/35

THE LINEAR SEQUENCE OF FOURTHS

4/3 35/24 4/3 81/56 4/3 4/3

MODULATED SEQUENCE 1

2	3	4	5	6	7	1
6/5	4/3	35/24	4/3	81/56	4/3	4/3

NEW SCALE 1

1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1
9/8 · 28/27 · 36/35 · 5/4 · 28/27 · 36/35 · 5/4

MODULATED SEQUENCE 2

3	2	4	5	6	7	1
4/3	**6/5**	35/24	4/3	81/56	4/3	4/3

NEW SCALE 2

1/1 9/8 7/6 4/3 3/2 14/9 8/5 2/1
9/8 · 28/27 · 8/7 · 9/8 · 28/27 · 36/35 · 5/4

original scale is restored, but transposed to a new and remote key. Wilson also provides an alternate derivation which better brings out the transpositional character of the process. In this case the linear sequence of non-exceptional fourths is tandemly duplicated to form a series of indefinite extent. Successive overlapping 6-unit segments of this series are appended with the exceptional fourth to form octave scales. After seven operations, the sequence repeats with a new mode of the original scale. The process is illustrated in 6-16.

Non-traditional scale forms

In the remainder of this chapter, some non-traditional approaches to scale construction from tetrachordal modules will be presented. These approaches are presented as alternatives to the historical modes and other types of scales which were discussed in the earlier parts of this chapter.

The first group of non-standard tetrachordal scales is generated by combining a given tetrachord with an identical one transposed by one of its own structural intervals or the inversion of one of these intervals (6-17). This process yields 7-tone scales, including three of the traditional modes if the interval is 4/3, 3/2, or with a slight stretching of the concept, 9/8 and 3/2 together. The other tetrachordal complexes, however, are quite different from the historical modes.

6-17. *Complexes of one tetrachordal form.*

1. TRANSPOSITION BY *a*
1/1 *a b* 2*a ab* 4/3 4*a*/3 2/1

2. TRANSPOSITION BY *b*
1/1 *a b ab* 2*b* 4/3 4*b*/3 2/1

3. TRANSPOSITION BY 4/3, MIXOLYDIAN
1/1 *a b* 4/3 4*a*/3 4*b*/3 16/9 2/1

4. TRANSPOSITION BY 3/2, DORIAN
1/1 *a b* 4/3 3/2 3*a*/2 3*b*/2 2/1

5. TRANSPOSITION BY 2/*b*
1/1 *a b* 4/3 2/*b a*/*b* 4/3*b* 2/1

6. TRANSPOSITION BY 2/*a*
1/1 *a b* 4/3 2/*a b*/*a* 4/3*a* 2/1

7. TRANSPOSITION BY 9/8 & 3/2, HYPODORIAN
1/1 9/8 9*a*/8 9*b*/8 3/2 3*a*/2 3*b*/2 2/1

8. TRANSPOSITION BY 4/3*b*
1/1 *a b* 4/3*b* 4/3 4*a*/3*b* 16/9*b* 2/1

9. TRANSPOSITION BY 4/3*a*
1/1 *a b* 4/3*a* 4/3 4*b*/3*a* 16/9*a* 2/1

10. TRANSPOSITION BY *a*/*b*
1/1 *a*2/*b a b* 4*a*/3*b* 4/3 *a*/*b* 2/1

11. TRANSPOSITION BY *b*/*a*
1/1 *b*/*a a b b*2/*a* 4/3 4*b*/3*a* 2/1

6-18 provides examples of the resulting scales when the generating tetrachord is Archytas's enharmonic, 28/27 · 36/35 · 5/4. In this case interval *a* equals 28/27 and *b* is 16/15 (28/27·36/35).

As some of these tetrachordal complexes have large gaps, one might try combining two of them, one built upwards from 1/1 and the other downwards from 2/1 to create a more even scale, though there are precedents for such gapped scales, i.e., the Mixolydian harmonia (6-4). While the normal ascending or prime form of the tetrachord—the one whose intervals are in the order of smallest, medium and largest—is used to demonstrate the technique, any of the six permutations would serve equally well. In fact, Archytas's enharmonic and diatonic genera are not strictly of this form as 28/27 is larger than 36/35 and 8/7 is wider than 9/8.

The next class of tetrachordal complexes are those composed of a tetrachord and its inverted form. 6-19 lists some simple examples of this approach; 6-20 lists the resulting notes in Archytas's enharmonic tuning. These scales have six, seven, or eight tones.

6-18. *Complexes of the prime form of Archytas's enharmonic.*

1. TRANSPOSITION BY *a*
1/1 28/27 16/15 784/729 448/405 4/3 112/81 2/1
0 63 112 126 175 498 561 1200

2. TRANSPOSITION BY *b*
1/1 28/27 16/15 448/405 256/225 4/3 64/32 2/1
0 63 112 175 223 498 610 1200

3. TRANSPOSITION BY 4/3 MIXOLYDIAN
1/1 28/27 16/15 4/3 112/81 64/45 16/9 2/1
0 63 112 498 561 610 996 1200

4. TRANSPOSITION BY 3/2, DORIAN
1/1 28/27 16/15 4/3 3/2 14/9 8/5 2/1
0 63 112 498 702 765 814 1200

5. TRANSPOSITION BY 2/*b*
1/1 28/27 16/15 5/4 4/3 15/8 35/18 2/1
0 63 112 386 498 1088 1151 1200

6. TRANSPOSITION BY 2/*a*
1/1 36/35 28/27 16/15 9/7 4/3 27/14 2/1
0 49 63 112 435 498 1137 1200

7. TRANSPOSITION BY 9/8 & 3/2, HYPODORIAN
1/1 9/8 7/6 6/5 3/2 14/9 8/5 2/1
0 204 267 316 702 765 814 1200

8. TRANSPOSITION BY 4/3*b*
1/1 28/27 16/15 5/4 35/27 4/3 5/3 2/1
0 63 112 386 449 498 884 1200

9. TRANSPOSITION BY 4/3*a*
1/1 28/27 16/15 9/7 4/3 48/35 12/7 2/1
0 63 112 435 498 547 933 1200

10. TRANSPOSITION BY *a/b*
1/1 245/243 28/27 16/15 35/27 4/3 35/18 2/1
0 14 63 112 449 498 1151 1200

11. TRANSPOSITION BY *b/a*
1/1 36/35 28/27 16/15 192/175 4/3 48/35 2/1
0 49 63 112 161 498 561 1200

6-19. *Simple complexes of prime and inverted forms. Two versions of the pseudo- (Ψ-) Hypodorian mode are shown to illustrate the effect of reversing the placement of the prime and inverted forms. The two scales are not modes of each other.*

1. TRANSPOSITION AND INVERSION BY *a*, 6 TONES, A HEXANY
1/1 *a b* 4*a*/3*b* 4/3 4*a*/3 2/1

2. TRANSPOSITION AND INVERSION BY *b*, 6 TONES, A HEXANY
1/1 *a b* 4/3 4*b*/3*a* 4*b*/3 2/1

3. TRANSPOSITION AND INVERSION BY 4/3, 7 TONES, ψ-MIXOLYDIAN
1/1 *a b* 4/3 16/9*b* 16/9*a* 16/9 2/1

4. TRANSPOSITION AND INVERSION BY 3/2, 7 TONES, ψ-DORIAN
1/1 *a b* 4/3 3/2 2/*b* 2/*a* 2/1

5. TRANSPOSITION AND INVERSION BY 2/*b*, 8 TONES, AN OCTONY
1/1 *a b* 4/3 2/*b* 4/3b^2 4/3*ab* 4/3*b* 2/1

6. TRANSPOSITION AND INVERSION BY 2/*a*, 8 TONES, AN OCTONY
1/1 *a b* 4/3 2/*a* 4/3a^2 4/3*ab* 4/3*a* 2/1

7. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, ψ-HYPODORIAN 1
1/1 9/8 3/2*b* 3/2*a* 3/2 3*a*/2 3*b*/2 2/1

8. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, ψ-HYPODORIAN 2
1/1 9/8 9*a*/8 9*b*/8 3/2 2/*b* 2/*a* 2/1

9. TRANSPOSITION AND INVERSION BY 1/1, 6 TONES, A HEXANY
1/1 *a b* 4/3*b* 4/3*a* 4/3 2/1

10. TRANSPOSITION AND INVERSION BY 4/3*b*, 8 TONES, AN OCTONY
1/1 *a b* 4/3*b* 4/3 16/9b^2 16/9*ab* 16/9*b* 2/1

11. TRANSPOSITION AND INVERSION BY 4/3*a*, 8 TONES, AN OCTONY
1/1 *a b* 4/3*a* 4/3 16/9*ab* 16/9a^2 16/9*a* 2/1

12. TETRACHORDAL HEXANY, 6 TONES, A-MODE
1/1 *b*/*a b* 4/3*a* 4/3 4*b*/3*a* 2/1

13. EULER'S GENUS MUSICUM, 8 TONES, AN OCTONY
1/1 *a b ab* 4/3 4*a*/3 4*b*/3 4*ab*/3 2/1

14. TRANSPOSITION AND INVERSION BY B/A, 8 TONES, AN OCTONY
1/1 *b*/*a a b* 4/3*a* 4/3 4*b*/3*a*2 4*b*/3*a* 2/1

15. TRANSPOSITION AND INVERSION BY A/B, 8 TONES, AN OCTONY
1/1 *a b* 4*a*/3b^2 4/3*b* 4*a*/3*b* 4/3 *a*/*b* 2/1

6-20. *Simple complexes of the prime and inverted forms of Archytas's enharmonic, in ratios and cents. Two versions of the Ψ-hypodorian mode are shown to illustrate the effect of reversing the placement of the prime and inverted forms. The two scales are not modes of each other.*

The 7-tone scales are analogous to the traditional Greek modes, whose names are appropriated with a prefixed Ψ (for pseudo) to indicate their relationship to the prototypes. Although these 7-tone scales were produced by pairing a tetrachord with its inversion, in principle any two dissimilar permutations would yield a heptatonic scale. This degree of flexibility is not true of the 6- and 8-tone types for which the pairing of prime and inverted forms is mandatory.

1. TRANSPOSITION AND INVERSION BY *a*, 6 TONES, A HEXANY
1/1 28/27 16/15 35/27 4/3 112/81 2/1
0 63 112 449 498 561 1200

2. TRANSPOSITION AND INVERSION BY *b*, 6 TONES, A HEXANY
1/1 28/27 16/15 4/3 48/35 64/45 2/1
0 63 112 498 547 610 1200

3. TRANSPOSITION AND INVERSION BY 4/3, 7 TONES, Ψ-MIXOLYDIAN
1/1 28/27 16/15 4/3 5/3 12/7 16/9 2/1
0 63 112 498 884 933 996 1200

4. TRANSPOSITION AND INVERSION BY 3/2, 7 TONES, Ψ-DORIAN
1/1 28/27 16/15 4/3 3/2 15/8 27/14 2/1
0 63 112 498 702 1088 1137 1200

5. TRANSPOSITION AND INVERSION BY 2/*b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 75/64 135/112 5/4 4/3 15/8 2/1
0 63 112 275 323 386 498 1088 1200

6. TRANSPOSITION AND INVERSION BY 2/*a*, 8 TONES, AN OCTONY
1/1 28/27 16/15 135/112 243/196 9/7 4/3 27/14 2/1
0 63 112 323 372 435 498 1137 1200

7. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, Ψ-HYPODORIAN 1
1/1 9/8 45/32 81/56 3/2 14/9 8/5 2/1
0 204 590 639 702 765 814 1200

8. TRANSPOSITION AND INVERSION BY 9/8 & 3/2, 7 TONES, Ψ-HYPODORIAN 2
1/1 9/8 7/6 6/5 3/2 15/8 27/14 2/1
0 204 267 316 702 1088 1137 1200

9. TRANSPOSITION AND INVERSION BY 1/1, 6 TONES, A HEXANY
1/1 28/27 16/15 5/4 9/7 4/3 2/1
0 63 112 386 435 498 1200

10. TRANSPOSITION AND INVERSION BY 4/3*b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 5/4 4/3 25/16 45/28 5/3 2/1
0 63 112 386 498 773 821 884 1200

11. TRANSPOSITION AND INVERSION BY 4/3*a*, 8 TONES, AN OCTONY
1/1 28/27 16/15 9/7 4/3 45/28 81/49 12/7 2/1
0 63 112 435 498 821 870 933 1200

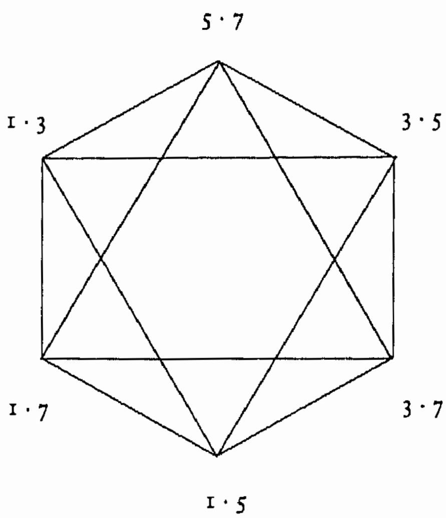
12. TETRACHORDAL HEXANY, 6 TONES, A-MODE
1/1 36/35 16/15 9/7 4/3 48/35 2/1
0 49 112 435 498 547 1200

13. EULER'S GENUS MUSICUM, 8 TONES, AN OCTONY
1/1 28/27 16/15 448/405 4/3 112/81 64/45 1792/1215 2/1
0 63 112 175 498 561 610 673 1200

14. TRANSPOSITION AND INVERSION BY *b/a*, 8 TONES, AN OCTONY
1/1 36/35 28/27 16/15 9/7 324/245 4/3 48/35 2/1
0 49 63 112 435 484 498 561 1200

15. TRANSPOSITION AND INVERSION BY *a/b*, 8 TONES, AN OCTONY
1/1 28/27 16/15 175/144 5/4 35/27 4/3 35/18 2/1
0 63 112 338 386 449 498 1151 1200

6-21. *The 1 3 5 7 tetradic hexany. The factor 1 may be omitted from the three tones which contain it. This diagram was invented by Ervin Wilson and represents the six tones of the hexany mapped over the six vertices of the regular octahedron (Wilson 1989). Each triangular face is an essential consonant chord of the hexany harmonic system and every pair of tones separated by a principal diagonal is a dissonance. The keynote is 3·5.*



NOTES AND INTERVALS OF HEXANY

1/1	7/6	4/3	7/5	8/5	28/15	2/1
7/6	8/7	21/20	8/7	7/6	15/14	
c	*b*	*a*	*b*	*c*	*d*	

Tetrachordal hexanies

The 6-tone complexes are of greater theoretical interest than either the seven or 8-tone scales. Because of their quasi-symmetrical melodic structure, which is a circular permutation of the interval sequence *c b a b c d* (*a*, *b*, *c*, and *d* not necessarily different intervals), they are members of a class of scales discovered by Ervin Wilson and termed *combination product sets* (Wilson 1989; Chalmers and Wilson 1982; Wilson, personal communication). The same structure results if interval *a* is replaced with interval *d* and intervals *b* and *c* are exchanged. A combination product set of six tones is called a *hexany* by Wilson.

The notes of the hexany are the melodic expansion of the intervals of a generating tetrad or tetrachord. They are obtained by forming the six binary products of the four elements of the generator. If these four elements are labelled x, y, z, and w, the resulting notes are $x \cdot y$, $x \cdot z$, $x \cdot w$, $y \cdot z$, $y \cdot w$, and $w \cdot z$. In the case where the generator is the dominant seventh tetrad, 1/1 5/4 3/2 7/4, written in factor form as 1 3 5 7, the resulting hexany is that of 6-21, where it has been mapped over the vertices of a regular octahedron. This diagram has been named a "hexagram" by Wilson.

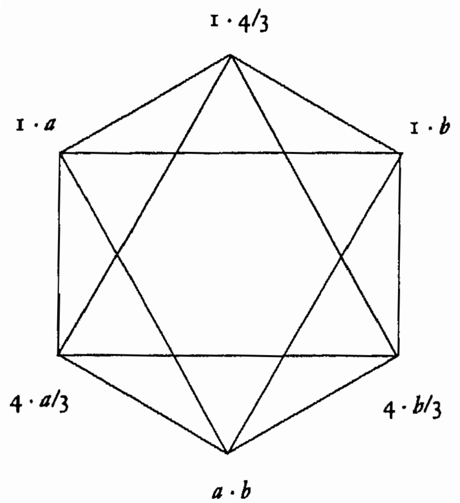
It is convenient to choose one of these tones and transpose the scale so that it starts on this note. The note $3 \cdot 5$ has been selected in 6-21. This note, however, should not be considered as the tonic of the scale; the combination product sets are harmonically symmetrical, polytonal sets with virtual or implicit tonics which are not tones of the scale. Although the hexany is partitionable into a set of rooted triads (see below), the global 1/1 for the whole set is not a note of the scale. In this sense, combination product sets are a type of atonal or non-centric musical structure in just intonation.

The four elements of the generator are related to the melodic intervals as $x = 1/1$, $y = b$, $z = b \cdot c$, and $w = a \cdot b^2 \cdot c$, although the actual tones may have to be transposed or circularly permuted to make this relationship clearer.

6-22. *Consonant chords of the 1 3 5 7 hexany.*

CHORD	HARMONIC	SUBHARMONIC
1 3 5	1·7 3·7 5·7	3·5 1·5 1·3
1 3 7	1·5 3·5 5·7	3·7 1·7 1·3
1 5 7	1·3 3·5 3·7	5·7 1·7 1·5
3 5 7	1·3 1·5 1·7	5·7 3·7 3·5

6-23. *The tetrachordal hexany. Based on the generating tetrad 1/1* a b *4/3. After transposition by* a, *it is equivalent to complex 12 of 6-19 and 6-20.*



NOTES AND INTERVALS OF HEXANY

1/1	*b/a*	*b*	4/3*a*	4/3	4*b*/3*a*	2/1
1/1	36/35	16/15	9/7	4/3	48/35	2/1

36/35	28/27	135/112	28/27	36/35	35/24
c	*b*	*a*	*b*	*c*	*d*

The six tones of the hexany may be partitioned into four sets of three tones and their inversions. In the hexagram or octahedral representation, the 3-tone sets appear as triangular faces or facets. The triads of 6-21 are tabulated in 6-22. These chords are the essential consonant chords of the hexany, and all chords containing pairs of tones separated by diagonals are considered dissonant.

Armed with this background, one can now proceed to the generation of hexanies from tetrachords. Starting with the tetrachord 1/1 *a b* 4/3 (the generator of complex 12 in 6-19), the generative process and the relationships between the notes may be seen in 6-23. Archytas's enharmonic (1/1 28/27 16/15 4/3; 28/27 · 36/35 · 5/4; *a* = 28/27, *b* = 16/15) is the specific generator (see also 6-20, complex 12). This hexany has been transposed so that the starting note 1·*a* is 1/1.

Tetrachordal hexanies are melodic developments of the basic intervals rather than harmonic expansions of tetrads. The triangular faces of tetrachordal hexanies are 2-interval subsets of the three intervals of the original tetrachord. Since this is basically a melodic development, the faces will be referred to as essential subsets rather than consonant chords. (For the same reason, the terms *harmonic* and *subharmonic* are replaced by *prime* and *inverted.*) These hexanies may be partitioned into essential subsets as shown in 6-24.

The generator of complex 1 of 6-19 and 6-20 (inversion and transposition by *a*) is the permuted tetrachord 1/1 *b/a b* 4/3 (1/1 36/35 16/15 4/3; 36/35 · 28/27 · 5/4; *a* = 36/35, *b* = 16/15). The generators of complexes 2 and 9 are 1/1 *b/a b* 4*b*/3*a* (1/1 36/35 16/15 48/35; 36/35 · 28/27 · 9/7) and

6-24. *Essential subsets of the hexanies based on the tetrachords 1/1* a b *4/3 and 1/1 28/27 16/15 4/3 (Archytas's enharmonic). For the sake of clarity, the factor 1 (1/1) has been omitted from 1·*a, *1·*b, *and 1·4/3. The · signs are also deleted. Both hexanies are given in their untransposed forms.*

SUBSET	PRIME	INVERTED
1/1 *a b*	4/3 4*a*/3 4*b*/3	*ab b a*
1/1 *a* 4/3	*b ab* 4*b*/3	4*a*/3 4/3 *a*
1/1 *b* 4/3	*a ab* 4*a*/3	4B/3 4/3 *b*
a b 4/3	*a b* 4/3	4*b*/3 4*a*/3 *ab*
1/1 28/27 16/15	4/3 112/81 64/45	448/405 16/15 28/27
1/1 28/27 4/3	16/15 448/405 64/45	112/81 4/3 28/27
1/1 16/15 4/3	28/27 448/405 112/81	64/45 4/3 16/15
28/27 16/15 4/3	28/27 16/15 4/3	64/45 112/81 448/405

1/1 *b/a b* 4/3*a* (1/1 36/35 16/15 9/7; 36/35 · 28/27 · 135/112) respectively. In these hexanies, the tetrachordal generators are bounded by augmented and diminished fourths rather than 4/3's, but the subset relations are analogous to those with perfect fourths.

Tetrachordal Euler genera

The 8-tone complexes represent a different type of scale which may be called an *interval symmetric set* (Chalmers and Wilson 1982; Chalmers 1983). These scales have the melodic sequence *d c b a b c d e* which is homologous to the *c b a b c d* sequence of the hexany. However, these 8-tone scales lack some of the harmonic and structural symmetries that characterize the combination product sets.

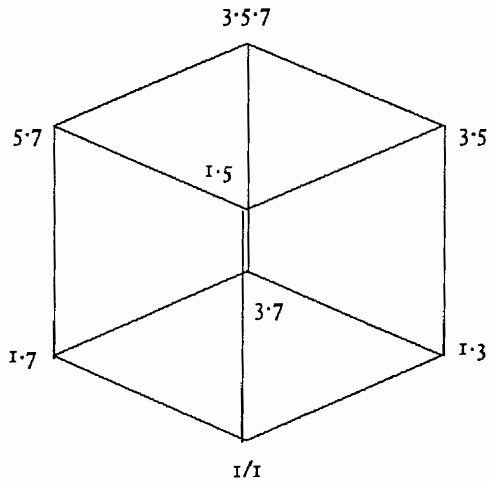
Wilson has pointed out that these sets are members of a large class of scales invented by Leonhard Euler in the eighteenth century and publicized by A. D. Fokker (Wilson, personal communication). While they have been given the generic name of octony in analogy with the hexany and other combination product sets, the terms Euler genus or Euler-Fokker genus would seem to have priority as collective names (Fokker 1966; Rasch 1987).

The generation of an octony from the 1 3 5 7 tetrad is shown in 6-25. In this representation, the eight tones have been mapped over the vertices of a cube. This diagram may be called an "octagram." The octony may also be partitioned into inversionally paired subsets, but the chords are generally more complex than those of hexanies derived from the same generator (6-26). Chords considered as the essential consonances of a harmonic system based on the octony appear not only as faces (face chords), but also as vertices with their three nearest neighbors connected by edges (vertex chords) or by face diagonals (vertex-diagonal chords) (Chalmers 1983). Essential dissonances are any chords containing a pair of tones separated by a principal diagonal of the cube.

With the exception of the generator itself and its inversion, each of the 4-note chords consists of the union of a harmonic and subharmonic triad of the form 1/1 *x y* and *x y x·y*. An analogous chord in traditional theory is the major triad with the major seventh added, 1/1 5/4 3/2 15/8, which could be construed as a major triad on 1/1 fused with a minor triad on 5/4.

As in the case of the hexany, octonies may be constructed from tetrachords and their inversions (6-27). The clearest example is complex 13 of

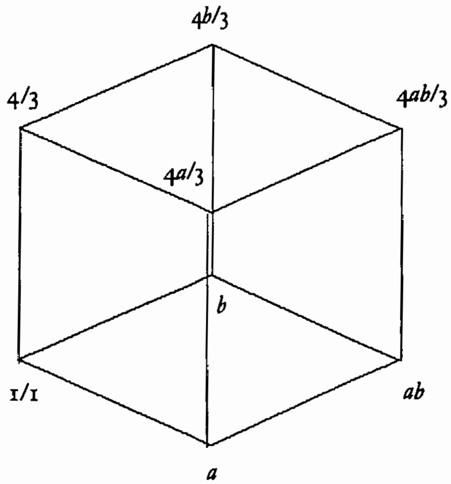
6-25. *The 1 3 5 7 tetradic octony. This structure is also an Euler's genus (Fokker 1966; Euler 1739).*



6-26. *Essential chords of the 1 3 5 7 tetradic octony.*

CHORD	PRIME	INVERTED
FACE	1/1 1·3 1·5 3·7	5·7 1·5 3·5 3·5·7
	1/1 1·5 1·5 3·5	1·7 5·7 3·7 3·5·7
	1/1 1·7 1·5 5·7	1·3 3·7 3·5 3·5·7
VERTEX	1/1 1·3 1·5 1·7	3·5·7 3·5 3·7 5·7
	1·7 5·7 1/1 3·7	1·5 1·3 3·5 3·5·7
	1·5 1/1 5·7 3·5	3·7 1·3 1·7 3·5·7
	1·3 3·5 3·7 1/1	5·7 1·7 1·5 3·5·7
DIAGONAL	1/1 5·7 3·5 3·7	3·5·7 1·5 1·3 1·7

6-27. *The tetrachordal octony. This 8-tone Euler's genus is generated from the generalized tetrachord* 1/1 a b *4/3.*



6-18 which is generated by the tetrachord 1/1 *a b* 4/3. Its subset structure is shown in 6-28. The generating tetrachord and its inversion appear as face chords. The other chords are more complex intervallic sets. Like the hexany above, the octony should be viewed as a melodic rather than a harmonic development of the tetrachord.

The other 8-tone complexes of 6-19 are also octonies. The complexes generated from Archytas's enharmonic genus are listed in 6-20.

Tetrachordal diamonds

The next group of non-traditional tetrachordal scales is even more complex than the previous constructions. The first of these are based on the Partch diamond (Partch [1949] 1974) which is an interlocking matrix of harmonic

NOTE AND INTERVALS OF OCTONY

1/1	*a*	*b*	*ab*	4/3	4*a*/3	4*b*/3	4*ab*/3	2/1
1/1	28/27	16/15	448/405	4/3	112/81	64/45	1792/1215	2/1

28/27	36/35	28/27	135/112	28/27	36/35	28/27	1215/896
d	*c*	*b*	*a*	*b*	*c*	*d*	*e*

6-28. *Essential subsets of the tetrachordal octonies 1/1* a b *4/3 and 1/1 28/27 16/15 4/3 (Archytas's enharmonic). The term* essential subset *rather than* consonant chord *is employed as the tetrachordal octony is primarily a melodic structure.*

SUBSET	PRIME	INVERTED
FACE	1/1 4/3 4*a*/3*a*	4*ab*/3 *ab b* 4*b*/3
	1/1 4/3 4*b*/3*b*	4*ab*/3 *ab a* 4*a*/3
	1/1 *a b ab*	4*ab*/3 4*b*/3 4*a*/3 4/3
VERTEX	1/1 *a b* 4/3	4*ab*/3 4*b*/3 4*a*/3 *ab*
	4/3 1/1 4*a*/3	4*b*/3 *ab* 4*ab*/3 *b a*
	4*a*/3 *a* 4/3	4*ab*/3*b* 4*b*/3 *ab* 1/1
	4*b*/3 4/3 *b*	4*ab*/3*a* *ab* 4*a*/3 1/1
DIAGONAL	1/1 4*b*/3 4*a*/3 *ab*	4*ab*/3 *a b* 4/3
FACE	1/1 4/3 112/81 28/27	1792/1215 448/405 16/15 64/45
	1/1 4/3 64/45 16/15	1792/1215 448/405 28/27 112/81
	1/1 28/27 16/15 448/405	1792/1215 64/45 112/81 4/3
VERTEX	1/1 28/27 16/15 4/3	1792/1215 64/45 112/81 448/405
	4/3 1/1 112/81 64/45	448/405 1792/1215 16/15 28/27
	112/81 28/27 4/3 16/15	64/45 448/405 1/1 1792/1215
	64/45 4/3 16/15 28/27	448/405 112/81 1/1 1792/1215
DIAGONAL	1/1 64/45 112/81 448/405	1792/1215 28/27 16/15 4/3

chords built on roots that are the elements of the corresponding subharmonic ones. An example of what is called a *5-limit* diamond may be seen in 6-30. This example has been constructed from harmonic 1 3 5; major triads and subharmonic 1 3 5; or minor triads. The structure is referred to as having a 5-limit because the largest prime number appearing among its ratios is five. Diamonds, however, may be constructed from any chord or scale of any cardinality, magnitude, or limit.

The simplest of the tetrachordal diamonds consists of ascending tetrachords erected on the notes of their inversions. Either the octave or the 4/3 (numbers 1 and 2 of 6-29) may be used as the interval of identity in the diamond. In the latter case, the resulting structure is one of the rare examples of musical scales in which the octave is not the interval of equivalence.

The second group of diamond-like complexes employs entire heptatonic scales in place of triads or tetrachords as structural elements. Four examples are given, all derived from scales of the Dorian or Ψ-Dorian type in which prime or inverted tetrachords appear in either or both positions relative to the central disjunctive tone (6-29, numbers 2, 4, 5; and 6-34). The prime-prime and inverted-inverted diamonds have prime or inverted tetrachords in both halves of the generating scales. Because of the inversional symmetry

6-29. *Tetrachordal diamonds. The octave modular tetrachordal diamond in Archytas's enharmonic tuning is shown in 6-33.*

1. THIRTEEN TONE OCTAVE MODULAR DIAMOND

1/1 *b/a a b* 4/3*b* 4/3*a* 4/3 3/2 3*a*/2 3*b*/2 2/*b* 2/*a a/b* 2/1

2. EIGHT TONE FOURTH MODULAR DIAMOND

1/1 *a b* 4/3*b* 4/3*a* 4*a*/3*b* 4/3 4*b*/3*a*

3. PRIME–PRIME AND INVERTED–INVERTED HEPTATONIC DIAMONDS, 27 TONES

1/1 *b/a a* B 9/8 9*a*/8 9*b*/8 4/3*b* 4/3*a* 4*a*/3*b* 4/3 4*b*/3*a* 4*a*/3 3/2*b* 4*b*/3 3/2*a* 3*a*/2*b* 3/2 3*b*/2*a* 3*a*/2 3*b*/2 16/9*b* 16/9*a* 16/9 2/*b* 2/*a a/b* 2/1

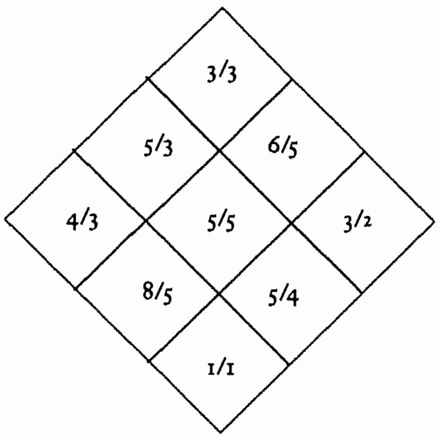
4. PRIME-INVERTED HEPTATONIC DIAMOND, 25 TONES

1/1 *b/a a b* a^2 *ab* 9/8 b^2 4/3*b* 4/3*a* 4/3 4*a*/3 3/2*b* 4*b*/33/2*a* 3/2 3*a*/2 3*b*/2 2/b^2 16/9 2/*ab* 2/a^2 2/*b* 2/*a a/b* 2/1

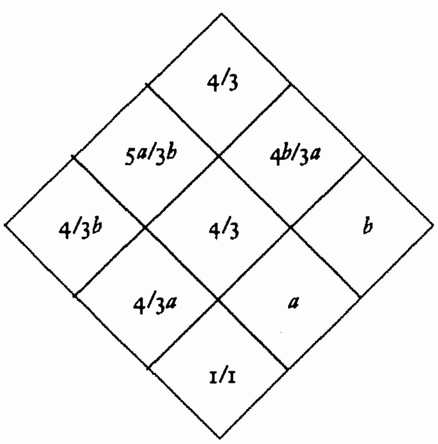
5. INVERTED-PRIME HEPTATONIC DIAMOND, 25 TONES

1/1 *b/a a b* 9/8 9*a*/8 9*b*/8 9a^2/8 9*ab*/8 4/3*b* 9b^2/8 4/3*a*4/3 3/2 3*a*/2 16/9b^2 3*b*/2 16/9*ab* 16/9a^2 16/9*b* 16/9*a* 16/92/*b* 2/*a a/b* 2/1

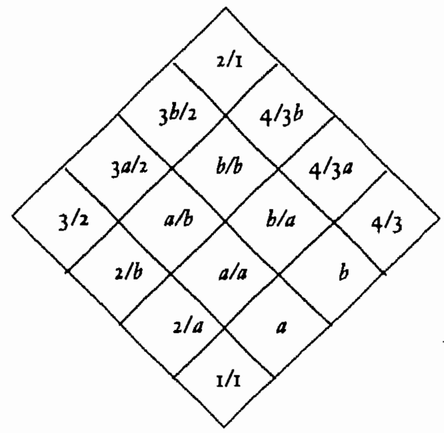
6-30. *Five-limit Partch diamond, after "The Incipient Tonality Diamond" (Partch [1949] 1974, 110). Based on the 1 3 5 major triad 1/1 5/4 3/2 and its inversion, the subharmonic 1 3 5 minor triad 2/1 8/5 4/3.*



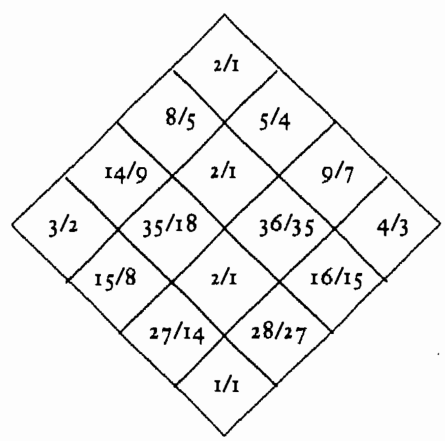
6-31. *Eight tone fourth modular diamond. Based on the tetrachord 1/1* a b *4/3, with 4/3 as the interval of equivalence.*



6-32. *Thirteen-tone octave modular tetrachordal diamond.*



6-33. *Thirteen-tone octave modular tetrachordal diamond based on Archytas's enharmonic genus.*



of the diamond, both scales are identical. The prime-inverted and inverted-prime diamonds are constructed from the corresponding tetrachordal forms and are non-equivalent scales, as in general, tetrachords are not inversionally symmetrical intervallic sequences. 6-35 and 6-36 show examples of these diamonds based on Archytas's enharmonic genus and its inversion.

6-34. *Tetrachordal heptatonic diamonds. These tables may be rotated 45 degrees clockwise to bring the diagonal of 2/1's into vertical position and compared to figures 6-30–33. The scale derived from the prime form of the tetrachord is seen in the rightmost column and its inversion in the bottom row.*

Stellated tetrachordal hexanies

The last of the non-traditional tetrachordal complexes to be discussed are two examples of stellated hexanies. Hexanies may be stellated by adding the eight tones which complete the partial tetrad or tetrachord on each face (Wilson 1989; Chalmers and Wilson 1982). The result is a complex of four

PRIME–PRIME

2/1	b/a	b	9b/8	3/2	3b/2a	3b/2
a/b	2/1	a	9a/8	3a/2b	3/2	3a/2
2/b	2/a	2/1	9/8	3/2b	3/2a	3/2
16/9b	16/9a	16/9	2/1	4/3b	4/3a	4/3
4/3	4b/3a	4b/3	3b/2	2/1	b/a	b
4a/3b	4/3	4a/3	3a/2	a/b	2/1	a
4/3b	4/3a	4/3	3/2	2/b	2/a	1/1

PRIME–INVERTED

2/1	b/a	4/3a	3/2a	2/ab	2/a2	2/a
a/b	2/1	4/3b	3/2b	2/b2	2/ab	2/b
3a/2	3b/2	2/1	9/8	3/2b	3/2a	3/2
4a/3	4b/3	16/9	2/1	4/3b	4/3a	4/3
ab	b2	4b/3	3b/2	2/1	b/a	b
a2	ab	4a/3	3a/2	a/b	2/1	a
a	b	4/3	3/2	2/b	2/a	1/1

INVERTED–INVERTED

2/1	b/a	4/3b	3/2a	3/2	3b/2a	2/a
a/b	2/1	4/3b	3/2b	3a/2b	3/2	2/b
3a/2	3b/2	2/1	9/8	9a/b	9b/a	3/2
4a/3	4b/3	16/9	2/1	a	b	4/3
4/3	4b/3a	16/9a	2/a	2/1	b/a	4/3a
4a/3b	4/3	16/9b	2/b	a/b	2/1	4/3b
a	b	4/3	3/2	3a/2	3b/2	1/1

INVERTED–PRIME

2/1	b/a	b	9b/8	9ab/8	9b2/8	3b/2
a/b	2/1	a	9a/8	9a2/8	9ab/8	3a/2
2/b	2/a	2/1	9/8	9a/8	9b/8	3/2
16/9b	16/9a	16/9	2/1	a	b	4/3
16/9ab	16/9a2	16/9a	2/a	2/1	b/a	4/3a
16/9b2	16/9ab	16/9b	2/b	a/b	2/1	4/3b
4/3b	4/3a	4/3	3/2	3a/2	3b/2	1/1

prime and four inverted tetrachords with a total of fourteen tones, though certain genera may produce degenerate complexes with fewer than 14 different notes. Wilson has variously termed these structures "mandalas" from their appearance in certain projections, and "tetradekanies" or "dekatesseranies" from their fourteen tones. Their topology is that of Kepler's stella octangula, an 8-pointed star-polyhedron (Coxeter 1973; Cundy and Rollett 1961).

The prime form of the tetrachord 1/1 *a b* 4/3 generates the hexany tones *a*, *b*, 4/3, 4*a*/3, 4*b*/3 and *ab* (*a* = 1/1·*a* or 1·*a*, etc.). This hexany is equivalent

6-35. *Tetrachordal diamonds based on Archytas's enharmonic, in ratios and cents.*

13-TONE OCTAVE MODULAR DIAMOND

1/1	36/35	28/27	16/15	5/4	9/7	4/3	3/2	14/9	8/5	15/8	27/14	35/18	2/1
0	49	63	112	386	435	498	702	765	814	1088	1137	1151	1200

8-TONE TETRACHORD MODULAR DIAMOND

1/1	28/27	16/15	5/4	9/7	35/27	4/3	48/35
0	63	112	386	435	449	498	547

PRIME-PRIME AND INVERTED-INVERTED HEPTATONIC DIAMONDS, 27 TONES

1/1	36/35	28/27	16/15	9/8	7/6	6/5	5/4	9/7	35/27	4/3	48/35	112/81		
0	49	63	112	204	267	316	386	435	449	498	547	561		
45/32	64/45	81/56	35/24	3/2	54/35	14/9	8/5	5/3	12/7	16/9	15/8	27/14	35/18	2/1
590	610	639	653	702	751	765	814	884	933	996	1088	1137	1151	1200

PRIME-INVERTED HEPTATONIC DIAMOND, 25 TONES

1/1	36/35	28/27	16/15	784/729	448/405	9/8	256/225	5/4	9/7	4/3	112/81		
0	49	63	112	126	175	204	223	386	435	498	561		
45/32	64/45	81/56	3/2	14/9	8/5	225/128	16/9	405/224	729/392	15/8	27/14	35/18	2/1
590	610	639	702	765	814	977	996	1025	1074	1088	1137	1151	1200

INVERTED-PRIME HEPTATONIC DIAMOND, 25 TONES

1/1	36/35	28/27	16/15	9/8	7/6	6/5	98/81	56/45	5/4	32/25			
0	49	63	112	204	267	316	330	379	386	427			
9/7	4/3	3/2	14/9	25/16	8/5	45/28	81/49	5/3	12/7	15/8	27/14	35/18	2/1	16/9
435	498	702	765	773	814	821	870	884	933	996	1088	1137	1151	1200

6-36. *Tetrachordal heptatonic diamonds based on Archytas's enharmonic. The generating tetrachords are 1/1 5/4 9/7 4/3 and 1/1 28/27 16/15 4/3.*

PRIME–PRIME

2/1	36/35	16/15	6/5	3/2	54/35	8/5
35/18	2/1	28/27	7/6	35/24	3/2	14/9
15/8	27/14	2/1	9/8	45/32	81/56	3/2
5/3	12/7	16/9	2/1	5/4	9/7	4/3
4/3	48/35	64/45	8/5	2/1	36/35	16/15
35/27	4/3	112/81	14/9	35/18	2/1	28/27
5/4	9/7	4/3	3/2	15/8	27/14	1/1

PRIME–INVERTED

2/1	36/35	9/7	81/56	405/224	729/392	27/14
35/18	2/1	5/4	45/32	225/128	405/224	15/8
14/9	8/5	2/1	9/8	45/32	81/56	3/2
112/81	64/45	16/9	2/1	5/4	9/7	4/3
448/405	256/225	64/45	8/5	2/1	36/35	16/15
784/729	448/405	112/81	14/9	35/18	2/1	28/27
28/27	16/15	4/3	3/2	15/8	27/14	1/1

INVERTED–INVERTED

2/1	36/35	9/7	81/56	3/2	54/35	27/14
35/18	2/1	5/4	45/32	35/24	3/2	15/8
14/9	8/5	2/1	9/8	7/6	6/5	3/2
112/81	64/45	16/9	2/1	28/27	16/15	4/3
4/3	48/35	12/7	27/14	2/1	36/35	9/7
35/27	4/3	5/3	15/8	35/18	2/1	5/4
28/27	16/15	4/3	3/2	14/9	8/5	1/1

INVERTED–PRIME

2/1	36/35	16/15	6/5	56/45	32/25	8/5
35/18	2/1	28/27	7/6	98/81	56/45	14/9
15/8	27/14	2/1	9/8	7/6	6/5	3/2
5/3	12/7	16/9	2/1	28/27	16/15	4/3
45/28	81/49	12/7	27/14	2/1	36/35	9/7
25/16	45/28	5/3	15/8	35/18	2/1	5/4
5/4	9/7	4/3	3/2	14/9	8/5	1/1

6-37. *Stellated hexanies generated by the prime tetrachord 1/1* a b *4/3. The hexany notes are* a, b, *4/3,* ab, *4*a/*3, and* 4b/3. *The 8 extra notes are (1/1)2=1/1,* a^2, b^2, *16/9, 3*ab/2, *4*ab/*3, 4*a/*3*b, *and 4*b/*3*a. *The second stellated hexany is based on number 1 of figure 6-29. Instances of each are based on Archytas's enharmonic. The first is generated by prime tetrachord 1/1 28/27 16/15 4/3. The hexany notes are 28/27, 16/15, 4/3, 448/405, 112/81, and 64/45. The second is based on (1) of 6-20.*

FIRST STELLATED TETRACHORDAL HEXANY

1/1	*a*	*b*	a^2	*ab*	b^2	4*a*/3*b*	4/3	4*b*/3*a*	4*a*/3	4*b*/3	4*ab*/3	3*ab*/2	16/9	2/1
1/1	28/27	16/15	784/729	448/405	256/225	35/27	4/3	48/35	112/81	64/45	1792/1215	224/135	16/9	2/1
0	63	112	126	175	223	449	498	547	561	610	673	877	996	1200

SECOND STELLATED TETRACHORDAL HEXANY

1/1	*b*/*a*	b^2/a^2	*b*	b^2/a	b^2	4/3*a*	4/3	4*b*/3*a*	4*a*/3	4*b*/3	$4b^2/3a$	$3b^2/2a$	16/9	2/1
1/1	36/35	1296/1225	16/15	192/175	256/225	9/7	4/3	48/35	112/81	64/45	256/175	288/175	16/9	2/1
0	49	98	112	161	223	435	498	547	561	610	659	862	996	1200

to complex 12 of 6-19 when transposed so as to begin on the tone *a*. The stellated form of this hexany is the first of 6-37, while complex 1 of 6-19 yields the second of 6-37. The eight supplementary tones of the first stellated hexany are 1/1, a^2, b^2, 16/9, 4*a*/3*b*, 4*ab*/3, 3*ab*/2, and 4*b*/3*a*. These notes may be deduced by inspection of 6-23, the tetrachordal hexany. The first four extra notes are the squares of the elements of the generator, 1/1, a^2, b^2, and 16/9 (x^2, y^2, z^2, and w^2) from 1/1 *a b* and 4/3. The remaining four notes are the mixed product-quotients needed by the subharmonic faces. These have the form $x{\cdot}y{\cdot}z/w$ (3*ab*/2), $x{\cdot}y{\cdot}w/z$ (4*a*/3*b*), $x{\cdot}z{\cdot}w/y$ (4*b*/3*a*), and $y{\cdot}z{\cdot}w/x$ (4*ab*/3). Two stellated hexanies based on Archytas's enharmonic are shown in 6-37.

The notes of the second type of stellated hexany of 6-30 are derived analogously by replacing *a* in the prime tetrachord with *b*/*a*. The tetrachord 1/1 28/27 16/15 4/3 in the first type is thus replaced by 1/1 36/35 16/15 4/3.

The essential tetrachords of the first stellated hexany are seen in 6-38, and those of the second may be found by analogy. The component tetrachords of the first stellated hexany derived from Archytas's enharmonic are listed in 6-39. Those of the second kind may be derived by replacing the 28/27 of the first tetrachord with 36/35. The other tetrachordal hexanies of 6-18 also generate stellated hexanies, but their tetrachords are bounded by intervals other than 4/3.

6-38. (a) *Essential tetrachords of the first stellated hexany. For the sake of clarity, the factor 1 (1/1) has been omitted from 1 · a, 1 · b, 1 · 4/3, etc. The · signs are also deleted. The boldfaced notes in each chord are the starting notes of the prime and inverted tetrachords, 1/1* a b *4/3 and 4/3 4/3*a *4/3*b *1/1.*

PRIME				INVERTED			
1/1	*a*	*b*	4/3	4/3	4/3*a*	4/3*b*	**1/1**
4/3	4*a*/3	4*b*/3	16/9	*ab*	*b*	*a*	**3*ab*/2**
b	*ab*	*b*2	4*b*/3	4*a*/3	4/3	4*a*/3*b*	***a***
a	*a*2	*ab*	4*a*/3	4*b*/3	4*b*/3*a*	4/3	***b***
1/1	*a*	*b*	4/3	4*ab*/3	4*b*/3	4*a*/3	***ab***

6-39. *Essential tetrachords of the 1/1 28/27 16/15 4/3 stellated hexany.*

PRIME				INVERTED			
1/1	28/27	16/15	4/3	4/3	9/7	5/4	**1/1**
4/3	112/81	64/45	16/9	448/405	16/15	28/27	**224/135**
16/15	448/405	256/225	64/45	112/81	4/3	35/27	**28/27**
28/27	784/729	448/405	112/81	64/45	48/35	4/3	**16/15**
1/1	28/27	16/15	4/3	1792/1215	64/45	112/81	**448/405**