3 Aristoxenos and the geometrization of musical space

ARISTOXENOS WAS FROM the Greek colony of Tarentum in Italy, the home of the famous musician and mathematician Archytas. In the early part of his life, he was associated with the Pythagoreans, but in his later years he moved to Athens where he studied under Aristotle and absorbed the new logic and geometry then being developed (Barbera 1980; Crocker 1966; Litchfield 1988). He was the son of the noted musician Spintharos, who taught him the conservative musical tradition still practiced in the Greek colonies, if not in Athens itself (Barbera 1978).

The geometry of music

The new musical theory that Aristoxenos created about 320 BCE differed radically from that of the Pythagorean arithmeticians. Instead of measuring intervals with discrete ratios, Aristoxenos used continuously variable quantities. Musical notes had ranges and tolerances and were modeled as loci in a continuous linear space. Rather than ascribing the consonance of the octave, fifth, and fourth to the superparticular nature of their ratios, he took their magnitude and consonance as given. Since these intervals could be slightly mistuned and still perceived as categorically invariant, he decided that even the principal consonances of the scale had a narrow, but still acceptable range of variation. Thus, the ancient and bitter controversy over the allegedly unscientific and erroneous nature of his demonstration that the perfect fourth consists of two and one half tones is really inconsequential.

Aristoxenos defined the whole tone as the difference between the two fundamental intervals of the fourth and the fifth, the only consonances smaller than the octave. The octave was found to consist of a fourth and a fifth, two fourths plus a tone, or six tones. The intervals smaller than the fourth could have any magnitude in principle since they were dissonances and not precisely definable by the unaided ear, but certain sizes were traditional and distinguished the genera known to every musician. These conventional intervals could be measured in terms of fractional tones by the ear alone because musical function, not numerical precision, was the criterion. The tetrachords that Aristoxenos claimed were well-known are shown in 3-1.

Aristoxenos described his genera in units of twelfths of a tone (Macran 1902), but later theorists, notably Cleonides, translated these units into a cipher consisting of 30 parts (*moria*) to the fourth (Barbera 1978). The enharmonic genus consisted of a pyknon divided into two 3-part microtones or *dieses* and a ditone of 24 parts to complete the perfect fourth. Next come three shades of the chromatic with dieses of 4, 4.5, and 6 parts and upper intervals of 22, 21, and 18 parts respectively. The set was finished with two diatonic tunings, a soft diatonic (6 + 9 + 15 parts), and the intense diatonic (6 + 12 + 12 parts). The former resembles a chromatic genus, but the latter is similar to our modern conception of the diatonic and probably

3-1. *The genera of Aristoxenos. The descriptions of Aristoxenos (Macran 1902) in terms of twelfths of tones have been converted to cents, assuming 500 cents to the equally tempered fourth. The interpretation of Aristoxenos's fractional tones as thirty parts to the fourth is after the second century theorist Cleonides.*

ENHARMONIC

0 50 100 500

3 + 3 + 24 PARTS

1/4 + 1/4 + 2 TONES

50 + 50 + 400 CENTS

SOFT CHROMATIC

0 67 133 500

4 + 4 + 22 PARTS

1/3 + 1/3 + 1 5/6 TONES

67 + 67 + 333 CENTS

HEMIOLIC CHROMATIC

0 75 150 500

4.5 + 4.5 + 21 PARTS

3/8 + 3/8 + 1 3/4 TONES

75 + 75 + 350 CENTS

INTENSE CHROMATIC

0 100 200 500

6 + 6 + 18 PARTS

1/2 + 1/2 + 1 1/2 TONES

100 + 100 + 300 CENTS

SOFT DIATONIC

0 100 250 500

6 + 9 + 15 PARTS

1/2 + 3/4 + 1 1/4 TONES

100 + 150 + 250 CENTS

INTENSE DIATONIC

0 100 300 500

6 + 12 + 12 PARTS

1/2 + 1 + 1 TONES

100 + 200 + 200 CENTS

3-2. *Other genera mentioned by Aristoxenos.*

UNNAMED CHROMATIC

0 67 200 500

4 + 8 + 18 PARTS

1/3 + 2/3 + 1 1/2 TONES

67 + 133 + 300 CENTS

DIATONIC WITH SOFT CHROMATIC DIESIS

0 67 300 500

4 + 14 + 12 PARTS

1/3 + 1 1/6 + 1 TONES

67 + 233 + 200 CENTS

DIATONIC WITH HEMIOLIC CHROMATIC DIESIS

0 75 300 500

4.5 + 13.5 + 21 PARTS

3/8 + 1 1/8 + 1 TONES

75 + 225 + 200 CENTS

REJECTED CHROMATIC

0 100 150 500

6 + 3 + 21 PARTS

1/2 + 1/4 + 1 3/4 TONES

100 + 50 + 350 CENTS

UNMELODIC CHROMATIC

0 75 133 500

4.5 + 3.5 + 22 PARTS

3/8 + 7/24 + 1 5/6 TONES

75 + 58 + 367 CENTS

represents the Pythagorean form. Two such 30-part tetrachords and a whole tone of twelve parts completed an octave of 72 parts.

Several properties of the Aristoxenian tetrachords are immediately apparent. The enharmonic and three chromatic genera have small intervals with similar sizes, as if the boundary between the enharmonic and chromatic genus was not yet fixed. The two chromatics between the syntonic chromatic and the enharmonic may represent developments of neutral-third pentatonics mentioned in chapter 2.

The pyknon is always divided equally except in the two diatonic genera whose first intervals (half tones) are the same as that of the syntonic chromatic. Thus Aristoxenos is saying that the first interval must be less than or equal to the second, in agreement with Ptolemy's views nearly five hundred years later.

The tetrachords of 3-2 are even more interesting. The first, an approved but unnamed chromatic genus, not only has the 1:2 division of the pyknon, but more importantly, is extremely close to Archytas's chromatic tuning (Winnington-Ingram 1932). The diatonic with soft chromatic diesis is a very good approximation to Archytas's diatonic as well (ibid.). Only Archytas's enharmonic is missing, though Aristoxenos seems to allude to it in his polemics against raising the second string and thus narrowing the largest interval (ibid.). These facts clearly show that Aristoxenos understood the music of his time.

The last two tetrachords in 3-2 were considered unmusical because the second interval is larger than the first. Winnington-Ingram (1932) has suggested that Aristoxenos could have denoted Archytas's enharmonic tuning as 4 + 3 + 23 parts (67 + 50 + 383), a tuning which suffers from the same defect as the two rejected ones. A general prejudice against intervals containing an odd number of parts may have caused Aristoxenos to disallow tetrachords such as 5 + 11 + 14, 5 + 9 + 16 (ibid.), and 5 + 6 + 19 (Macran 1902).

The alleged discovery of equal temperament

Because a literal interpretation of Aristoxenos's parts implies equal temperaments of either 72 or 144 tones per octave to accommodate the hemiolic chromatic and related genera, many writers have credited him with the discovery of the traditional western European 12-tone intonation. This conclusion would appear to be an exaggeration, at the least. There is no evidence whatsoever in any of Aristoxenos's surviving writings or from any of the later authors in his tradition that equal temperament was intended (Litchfield 1988).

Greek mathematicians would have had no difficulty computing the string lengths for tempered scales, especially since only two computations for each tetrachord would be necessary, and only a few more for the complete octave scale. Methods for the extraction of the square and cube roots of two were long known, and Archytas, the subject of a biography by Aristoxenos, was renowned for having discovered a three-dimensional construction for the cube root of two, a necessary step for dividing the octave into the 12, 24, 36, 72, or 144 geometric means as required by Aristoxenos's tetrachords (Heath [1921] 1981, 1:246-249). Although irrationals were a source of great worry to Pythagorean mathematicians, by Ptolemy's time various mechanical instruments such as the *mesolabium* had been invented for extracting roots and constructing geometric means (ibid., 2:104). Yet neither Ptolemy nor any other writer mentions equal temperament.

Ptolemy, in fact, utterly missed Aristoxenos's point and misinterpreted these abstract, logarithmic parts as aliquot segments of a real string of 120 units with 60 units at the octave, 80 at the fifth, and 90 at the fourth. His upper tetrachord had only twenty parts, necessitating the use of complicated fractional string lengths to express the actually simple relations in the upper tetrachords of the octave scales.

There are two obvious explanations for this situation. First, Aristoxenos was opposed to numeration, holding that the trained ear of the musician was sufficiently accurate. Second, Greek music was mostly monophonic, with heterophonic rather than harmonic textures. Although modulations and chromaticism did exist, they would not have demanded the paratactical pitches of a tempered gamut (Polansky 1987a). There was no pressing need for equal temperament, and if it was discovered, the fact was not recorded (for a contrary view, see McClain 1978).

Later writers and Greek notation

Although most of the later theorists continued the geometric approach taken by Aristoxenos, they added little to our knowledge of Greek music theory with few exceptions. Cleonides introduced the cipher of thirty parts to the fourth. Bacchios gave the names of some intervals of three and five dieses which were alleged to be features of the ancient style, and Aristides

3-3. *Two medieval Islamic forms. These two medieval Islamic tetrachords are Aristoxenian approximations to Ptolemy's equable diatonic. The Arabs also listed Aristoxenos's other tetrachords in their treatises.*

NEUTRAL DIATONIC

0 200 350 500

12 + 9 + 9 PARTS

1 + 3/4 + 3/4 TONES

200 + 150 + 150 CENTS

EQUAL DIATONIC

0 167 334 500

10 + 10 + 10 PARTS

5/6 + 5/6 + 5/6 TONES

167 + 167 + 166 CENTS

Quintilianus offered a purported list of the ancient harmoniai mentioned by Plato in the *Timaeus*.

One exception was Alypius, a late author who provided invaluable information on Greek musical notation. His tables of keys or *tonoi* were deciphered independently in the middle of the nineteenth century by Bellermann (1847) and Fortlage (1847), and made it possible for the few extant fragments of Greek music to be transcribed into modern notation and understood. Unfortunately, Greek notation lacked both the numerical precision of the tuning theories, and the clarity of the system of genera and modes (chapter 6). Additionally, there are unresolved questions concerning the choice of alternative, but theoretically equivalent, spellings of certain passages. Contemplation of these problems led Kathleen Schlesinger to the heterodox theories propounded in *The Greek Aulos*.

Others have simply noted that the notation and its nomenclature seem to have evolved away from the music they served until it became an academic subject far removed from musical needs (Henderson 1957). For these reasons, little will be said about notation; knowledge of it is not necessary to understand Greek music theory nor to apply Greek theory to present-day composition.

Medieval Islamic theorists

As the Roman empire decayed, the locus of musical science moved from Alexandria to Byzantium and to the new civilization of Islam. Aristoxenos's geometric tradition was appropriated by both the Greek Orthodox church to describe its liturgical modes. Aristoxenian doctrines were also included in the Islamic treatises, although arithmetic techniques were generally employed.

The tetrachords of 3-3 were used by Al-Farabi to express 3/4-tone scales similar to Ptolemy's equable diatonic in Aristoxenian terms. If one subtracts 10 + 10 + 10 parts from Ptolemy's string of 120 units, one obtains the series 120 110 100 90, which are precisely the string lengths for the equable diatonic (12/11 · 11/10 · 10/9). It would appear that the nearly equal tetrachord 11/10 · 11/10 · 400/363 was not intended.

The tetrachord 12 + 9 + 9 yields the permutation 120 108 99 90, or 10/9 · 12/11 · 11/10. This latter tuning is similar to others of Al-Farabi and Avicenna consisting of a tone followed by two 3/4-tone intervals. Other tetrachords of this type are listed in the Catalog.

Eastern Orthodox liturgical music

The intonation of the liturgical music of the Byzantine and Slavonic Orthodox churches is a complex problem and different contemporary authorities offer quite different tunings for the various scales and modes (*echoi*). One of the complications is that until recently a system of 28 parts to the fourth, implying a 68-note octave (28 + 12 + 28 = 68 parts), was in use along with the Aristoxenian 30 + 12 + 30 parts (Tiby 1938).

Another problem is that the nomenclature underwent a change; the term enharmonic was applied to both a neo-chromatic and a diatonic genus, and chromatic was associated with the neo-chromatic forms. Finally, many of the modes are composed of two types of tetrachord, and both chromaticism and modulation are commonly employed in melodies.

Given these complexities, only the component tetrachords extracted from the scales are listed in 3-4. The format of this table differs from that of 3-1 through 3-3 in that the diagrams have been omitted and partially replaced by the ratios of plausible arithmetic forms. The four tetrachords from Tiby which utilize a system of 28 parts to the fourth are removed to the Tempered section of the Catalog.

3-4. *Byzantine and Greek Orthodox tetrachords. Athanasopoulos's enharmonic and diatonic genera consist of various permutations of 6 + 12 + 12, i.e. 12 + 6 + 12. Xenakis permits permutations of the 12 + 11 + 7 and 6 + 12 + 12 genera. A closer, but non-superparticular, approximation to Xenakis's intense chromatic would be 22/21 · 6/5 · 35/33.*

PARTS	CENTS	RATIOS	GENUS
		ATHANASOPOULOS (1950)	
9 + 15 + 6	150 + 250 + 100	—	CHROMATIC
6 + 18 + 6	100 + 300 + 100	—	CHROMATIC
6 + 12 + 12	100 + 200 + 200	—	DIATONIC
12 + 12 + 6	200 + 200 + 100	—	ENHARMONIC
		SAVAS (1965)	
8 + 14 + 8	133 + 233 + 133	—	CHROMATIC
10 + 8 + 12	167 + 133 + 200	—	DIATONIC
8 + 12 + 10	133 + 200 + 167	—	BARYS DIATONIC
12 + 12 + 6	200 + 200 + 100	—	ENHARMONIC
8 + 16 + 6	133 + 267 + 100	—	BARYS ENHARMONIC
6 + 20 + 4	100 + 333 + 67	—	PALACE MODE (NENANO)
		XENAKIS (1971)	
7 + 16 + 7	117 + 266 + 117	16/15 · 7/6 · 15/14	SOFT CHROMATIC
5 + 19 + 6	83 + 317 + 100	256/243 · 6/5 · 135/128	INTENSE CHROMATIC
12 + 11 + 7	200 + 183 + 117	9/8 · 10/9 · 16/15	DIATONIC
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	ENHARMONIC

The tetrachords of Athanasopoulos (1950) are clearly Aristoxenian in origin and inspiration, despite being reordered. One of his chromatics is Aristoxenos's soft diatonic and the other is Aristoxenos's intense chromatic. The rest of his tetrachords are permutations of Aristoxenos's intense diatonic.

Savas's genera (Savas 1965) may reflect an Arabic or Persian influence, as diatonics with intervals between 133 and 167 cents are reminiscent of Al-Farabi's and Avicenna's tunings (chapter 2 and the Catalog). They may plausibly represent 12/11 and 11/10 so that his diatonic tunings are intended to approximate a reordered Ptolemy's equable diatonic. His chromatic resembles 14/13 · 8/7 · 13/12 and his Barys enharmonic, 15/14 · 7/6 · 16/15. Savas's ordinary enharmonic may stand for either Ptolemy's intense diatonic (10/9 · 9/8 · 16/15) or the Pythagorean version (256/243 · 9/8 · 9/8). The palace mode could be 15/14 · 6/5 · 28/27 (Ptolemy's intense chromatic). The above discussion assumes that some form of just intonation is intended.

The tunings of the experimental composer Iannis Xenakis (1971) are clearly designed to show the continuity of the Greek Orthodox liturgical tradition with that of Ptolemy and the other ancient arithmeticians, though they are expressed in Aristoxenian terms. This continuity is debatable; internal evidence suggests that the plainchant of the Roman Catholic church is derived from Jewish cantillation rather than Graeco-Roman secular music (Idelsohn 1921). It is hard to see how the music of the Eastern church could have had an entirely different origin, given its location and common early history. A case for evolution from a common substratum of Near Eastern music informed by classical Greek theory and influenced by the Hellenized Persians and Arabs could be made and this might give the appearance of direct descent.

The robustness of the geometric approach of Aristoxenos is still evident today after 2300 years. The musicologist James Murray Barbour, a strong advocate of equal temperament, proposed 2 + 14 + 14 and 8 + 8 + 14 as Aristoxenian representations of 49/48 · 8/7 · 8/7 and 14/13 · 13/12 · 8/7 in his 1953 book on the history of musical scales, *Tuning and Temperament*. With Xenakis's endorsement, Aristoxenian principles have become part of the world of international, or transnational, contemporary experimental music. In the next chapter the power of the Aristoxenian approach to generate new musical materials will be demonstrated.