2 Pythagoras, Ptolemy, and the arithmetic tradition

GREEK MUSICAL TRADITION begins in the sixth century BCE with the semi-legendary Pythagoras, who is credited with discovering that the frequency of a vibrating string is inversely proportional to its length. This discovery gave the Greeks a means to describe musical intervals by numbers, and to bring to acoustics the full power of their arithmetical science. While Pythagoras's own writings on music are lost, his tuning doctrines were preserved by later writers such as Plato, in the *Timaeus*, and Ptolemy, in the *Harmonics*. The scale derived from the *Timaeus* is the so-called Pythagorean tuning of Western European theory, but it is most likely of Babylonian origin. Evidence is found not only in cuneiform inscriptions giving the tuning order, but apparently also as music in a diatonic ***major mode*** (Duchesne-Guillemin 1963, 1969; Kilmer 1960; Kilmer et al. 1976). This scale may be tuned as a series of perfect fifths (or fourths) and octaves, having the ratios 1/1 9/8 81/64 4/3 3/2 27/16 243/128 2/1, though the Babylonians did not express musical intervals numerically.

The next important theorist in the Greek arithmetic tradition is Archytas, a Pythagorean from the Greek colony of Tarentum in Italy. He lived about 390 BCE and was a notable mathematician as well. He explained the use of the arithmetic, geometric, and harmonic means as the basis of musical tuning (Makeig 1980) and he named the ***harmonic mean***. In addition to his musical activities, he was renowned for having discovered a three-dimensional construction for the extraction of the cube root of two.

Archytas is the first theorist to give ratios for all three genera. His tunings are noteworthy for employing ratios involving the numbers 5 and 7 instead of being limited to the 2 and 3 of the orthodox Pythagoreans, for using the ratio 28/27 as the first interval (hypate to parhypate) in all three genera, and for employing the consonant major third, 5/4, rather than the harsher *ditone* 81/64, as the upper interval of the enharmonic genus. These tunings are shown in 2-1.

Other characteristics of Archytas's tunings are the smaller second interval of the enharmonic (36/35 is less than 28/27) and the complex second interval of his chromatic genus.

Archytas's enharmonic is the most consonant tuning for the genus, especially when its first interval, 28/27, is combined with a tone 9/8 below the tonic to produce an interval of 7/6. This note, called *hyperhypate*, is found not only in the *harmoniai* of Aristides Quintilianus (chapter 6), but also in the extant musical notation fragment from the first *stasimon* of Euripides's *Orestes*. It also occurs below a chromatic *pyknon* in the second Delphic hymn (Winnington-Ingram 1936). This usage strongly suggests that the second note of the enharmonic and chromatic genera was not a grace note as has been suggested, but an independent degree of the scale (ibid.). Bacchios, a much later writer, calls the interval formed by the skip from hyperhypate to the second degree an *ekbole* (Steinmayer 1985), further affirming the historical correctness of Archytas's tunings.

The complexity of Archytas's chromatic genus demands an explanation, as Ptolemy's soft chromatic *(chroma malakon)* 28/27 · 15/14 · 6/5 would seem to be more consonant. Evidently the chromatic pyknon still spanned the 9/8 at the beginning of the fourth century, and the 32/27 was felt to be

	ARCHYTAS'S GENERA	
28/27 · 36/35 · 5/4	63 + 49 + 386	ENHARMONIC
28/27 · 243/224 · 32/27	63 + 141 + 294	CHROMATIC
28/27 · 8/7 · 9/8	63 + 231 + 204	DIATONIC
	ERATOSTHENES'S GENERA	
40/39 · 39/38 · 19/15	44 + 45 + 409	ENHARMONIC
20/19 · 19/18 · 6/5	89 + 94 + 316	CHROMATIC
256/243 · 9/8 · 9/8	90 + 204 + 204	DIATONIC
	DIDYMOS'S GENERA	
32/31 · 31/30 · 5/4	55 + 57 + 386	ENHARMONIC
16/15 · 25/24 · 6/5	112 + 71 + 316	CHROMATIC
16/15 · 10/9 · 9/8	112 + 182 + 204	DIATONIC

2-1. *Ptolemy's catalog of historical tetrachords, from the* Harmonics *(Wallis 1682). The genus 56/55 · 22/21 · 5/4 (31 + 81 + 386 cents) is also attributed to Ptolemy. Wallis says that this genus is in all of the manuscripts, but is likely to be a later addition. The statements of Avicenna and Bryennios that 46/45 is the smallest melodic interval supports this view.*

the proper tuning for the interval between the upper two tones. This may be in part because 32/27 makes a 4/3 with the *disjunctive tone* immediately following, but also because the melodic contrast between the 32/27 at the top of the tetrachord and the 7/6 with the hyperhypate below is not as great as the contrast between lower 7/6 and the upper 6/5 of Ptolemy's tuning.

Archytas's diatonic is also found among Ptolemy's own tunings (2-2) and appears in the *lyra* and *kithara* scales that Ptolemy claimed were in common practice in Alexandria in the second century CE. According to Winnington-Ingram (1932), it is even grudgingly admitted by Aristoxenos and thus would appear to have been the principal diatonic tuning from the fourth century BCE through the second CE, a period of some six centuries.

Archytas's genera represent a considerable departure from the austerity of the older Pythagorean forms:

ENHARMONIC: 256/243 · 81/64
CHROMATIC: 256/243 · 2187/2048 · 32/27
DIATONIC: 256/243 · 9/8 · 9/8

The enharmonic genus is shown as a *trichord* because the tuning of the enharmonic genus before Archytas is not precisely known. The semitone was initially undivided and may not have had a consistent division until the stylistic changes recorded in his tunings occurred. In other words, the *incomposite ditone*, not the incidental microtones, is the defining characteristic of the enharmonic genus.

The chromatic tuning is actually that of the much later writer Gaudentius (Barbera 1978), but it is the most plausible of the Pythagorean chromatic tunings.

The diatonic genus is the tuning associated with Pythagoras by all the authors from ancient times to the present (Winnington-Ingram 1932).

2-2. *Ptolemy's own tunings.*

46/45 · 24/23 · 5/4	38 + 75 + 386	ENHARMONIC
28/27 · 15/14 · 6/5	63 + 119 + 316	SOFT CHROMATIC
22/21 · 12/11 · 7/6	81 + 151 + 267	INTENSE CHROMATIC
21/20 · 10/9 · 8/7	85 + 182 + 231	SOFT DIATONIC
28/27 · 8/7 · 9/8	63 + 231 + 204	DIATONON TONIAION
256/243 · 9/8 · 9/8	90 + 204 + 204	DIATONON DITONIAION
16/15 · 9/8 · 10/9	112 + 204 + 182	INTENSE DIATONIC
12/11 · 11/10 · 10/9	151 + 165 + 182	EQUABLE DIATONIC

Ptolemy and his predecessors in Alexandria

In addition to preserving Archytas's tunings, Ptolemy (ca. 160 CE) also transmitted the tunings of Eratosthenes and Didymos, two of his predecessors at the library of Alexandria (2-1). Eratosthenes's (third century BCE) enharmonic and chromatic genera appear to have been designed as simplifications of the Pythagorean prototypes. The use of 40/39 and 20/19 for the lowest interval presages the remarkable *Tanbur of Baghdad* of Al-Farabi with its *subharmonic* division by the *modal determinant* 40 (Ellis 1885; D'Erlanger 1935) and some of Kathleen Schlesinger's speculations in *The Greek Aulos* (1939).

Didymos's enharmonic seems to be mere formalism; the enharmonic genus was extinct in music as opposed to theory by his time (first century BCE). His 1:1 *linear division* of the pyknon introduces the prime number 31 into the musical relationships and deletes the prime number 7, a change which is not an improvement harmonically, though it would be of less significance in a primarily melodic music. His chromatic, on the other hand, is the most consonant non-*septimal* tuning and suggests further development of the musical styles which used the chromatic genus. Didymos's diatonic is a permutation of Ptolemy's intense diatonic (diatonon syntonon). It seems to be transitional between the Pythagorean (*3-limit)* and *tertian* tunings.

Ptolemy's own tunings stand in marked contrast to those of his predecessors. In place of the more or less equal divisions of the pyknon in the genera of the earlier theorists, Ptolemy employs a roughly 1:2 melodic proportion. He also makes greater use of *superparticular* or *epimore* ratios than his forerunners; of his list, only the traditional Pythagorean diatonon ditoniaion contains *epimeres*, which are ratios of the form $(n + m) / n$ where $m > 1$.

The emphasis on superparticular ratios was a general characteristic of Greek musical theory (Crocker 1963; 1964). Only epimores were accepted even as successive consonances, and only the first epimores (2/1, 3/2, and 4/3) were permitted as simultaneous combinations.

There is some empirical validity to these doctrines: there is no question that the first epimores are consonant and that this quality extends to the next group, 5/4 and 6/5, else *tertian harmony* would be impossible. Consonance of the septimal epimore 7/6 is a matter of contention. To my ear, it is consonant, as are the epimeres 7/4 and 7/5 and the inversions of the epimores 5/4 and 6/5 (8/5 and 5/3). Moreover, Ptolemy noticed that octave compounds of consonances (which are not themselves epimores) were aurally consonant. It is clear, therefore, that it is not just the form of the ratio, but at least two factors, the size of the interval and the magnitude of the defining integers, that determines relative consonance. Nevertheless, there does seem to be some special quality of epimore ratios. I recall a visit to Lou Harrison during which he began to tune a harp to the tetrachordal scale 1/1 27/25 6/5 4/3 3/2 81/50 9/5 2/1. He immediately became aware of the non-superparticular ratio 27/25 by perceiving the lack of resonance in the instrument.

A complete list of all possible tetrachordal divisions containing only superparticular ratios has been compiled by I. E. Hofmann (Vogel 1975). Although the majority of these tetrachords had been discovered by earlier theorists, there were some previously unknown divisions containing very small intervals. The complete set is given in 2-3 and individual entries also appear in the Miscellaneous listing of the Catalog.

The equable diatonic has puzzled scholars for years as it appears to be an academic exercise in musical arithmetic. Ptolemy's own remarks rebut this interpretation as he describes the scale as sounding rather strange or foreign and rustic (ξενικοτερον μεν ποσ και αγροικοτερον, Winnington-Ingram 1932). Even a cursory look at ancient and modern Islamic scales from the Near East suggests that, on the contrary, Ptolemy may have heard a similar scale and very cleverly rationalized it according to the tenets of Greek theory. Such scales with 3/4-tone intervals may be related to

2-3. *Hofmann's list of completely superparticular divisions. This table has been recomposed after Hofmann from Vogel (1975). See Main Catalog for further information. (5) has also been attributed to Tartini, but probably should be credited to Pachymeres, a thirteenth-century Byzantine author.*

1.	256/255 · 17/16 · 5/4	NEW ENHARMONIC
2.	136/135 · 18/17 · 5/4	NEW ENHARMONIC
3.	96/95 · 19/18 · 5/4	WILSON'S ENHARMONIC
4.	76/75 · 20/19 · 5/4	AUTHOR'S ENHARMONIC
5.	64/63 · 21/20 · 5/4	SERRE'S ENHARMONIC
6.	56/55 · 22/21 · 5/4	PSEUDO-PTOLEMAIC ENHARMONIC
7.	46/45 · 24/23 · 5/4	PTOLEMY'S ENHARMONIC
8.	40/39 · 26/25 · 5/4	AVICENNA'S ENHARMONIC
9.	28/27 · 36/35 · 5/4	ARCHYTAS'S ENHARMONIC
10.	32/31 · 31/30 · 5/4	DIDYMOS'S ENHARMONIC
11.	100/99 · 11/10 · 6/5	NEW CHROMATIC
12.	55/54 · 12/11 · 6/5	BARBOUR
13.	40/39 · 13/12 · 6/5	BARBOUR
14.	28/27 · 15/14 · 6/5	PTOLEMY'S SOFT CHROMATIC
15.	16/15 · 25/24 · 6/5	DIDYMOS'S CHROMATIC
16.	20/19 · 19/18 · 6/5	ERATOSTHENES'S CHROMATIC
17.	64/63 · 9/8 · 7/6	BARBOUR
18.	36/35 · 10/9 · 7/6	AVICENNA
19.	22/21 · 12/11 · 7/6	PTOLEMY'S INTENSE CHROMATIC
20.	16/15 · 15/14 · 7/6	AL-FARABI
21.	49/48 · 8/7 · 8/7	AL-FARABI
22.	28/27 · 8/7 · 9/8	ARCHYTAS'S DIATONIC
23.	21/20 · 10/9 · 8/7	PTOLEMY'S SOFT DIATONIC
24.	14/13 · 13/12 · 8/7	AVICENNA
25.	16/15 · 19/18 · 10/9	PTOLEMY'S INTENSE DIATONIC
26.	12/11 · 11/10 · 10/9	PTOLEMY'S EQUABLE DIATONIC

Aristoxenos's hemiolic chromatic and may descend from neutral third pentatonics such as Winnington-Ingram's reconstruction of the *spondeion* or libation mode (Winnington-Ingram 1928 and chapter 6), if Sachs's ideas on the origin of the genera have any validity (Sachs 1943). In any case, the scale is a beautiful sequence of intervals and has been used successfully by both Harry Partch *(Windsong, Daphne of the Dunes)* and Lou Harrison, the latter in an improvisation in the early 1970s.

Ptolemy returned to the use of the number seven in his chromatic and soft diatonic genera and introduced ratios of eleven in his intense chromatic and equable diatonic. These tetrachords appear to be in agreement with the musical reality of the era, as most of the scales described as contemporary tunings for the lyra and kithara have septimal intervals (6-4).

Ptolemy's intense diatonic is the basis for Western European just intonation. The Lydian or C mode of the scale produced by this genus is the European major scale, but the *minor mode* is generated by the intervallic retrograde of this tetrachord, 10/9 · 9/8 · 16/15. This scale is not identical to the Hypodorian or A mode of 12-tone equally tempered, meantone, and Pythagorean intonations. (For further discussion of this topic, see chapters 6 and 7.)

The numerical technique employed by Eratosthenes, Didymos, and Ptolemy to define the majority of their tetrachords is called *linear division* and may be identified with the process known in Greek as *katapyknosis.* Katapyknosis consists of the division, or rather the filling-in, of a musical interval by multiplying its numerator and denominator by a set of integers of increasing magnitude. The resulting series of integers between the extreme terms generates a new set of intervals of increasingly smaller span as the multiplier grows larger. These intervals form a series of microtones which are then recombined to produce the desired melodic division, usually composed of epimore ratios. The process may be seen in 2-4 where it is applied to the enharmonic *pyknotic* interval 16:15. By extension, the pyknon may also be termed the katapyknosis (Emmanuel 1921). It consists of three notes, the *barypyknon,* or lowest note, the *mesopyknon,* or middle note, and the *oxypyknon,* or highest.

The harmoniai of Kathleen Schlesinger are the result of applying katapyknosis to the entire octave, 2:1, and then to certain of the ensuing intervals. In chapter 4 it is applied to the fourth to generate *indexed genera.*

The divisions of Eratosthenes and Didymos comprise mainly 1:1 divi-

2-4. *Genesis of the enharmonic pykna by katapyknosis. In principle, all pyknotic divisions can be generated by this process, although very high multipliers may be necessary in some cases. The ones shown are merely illustrative. See the Catalogs for the complete list. (1x) The basic form is the enharmonic trichord, or major third pentatonic, often ascribed to Olympos. (2x) Didymos's enharmonion, a "weak" form. (3x) Ptolemy's enharmonion, a "strong" form. To comply with Greek melodic canons, it was reordered as 46/45 · 24/23 · 5/4. (4x) Serre's enharmonic, sometimes attributed to Tartini, and discussed by Perrett (1926, 26). Pachymeres may be the earliest source. (5x) Author's enharmonic, also on Hofmann's list of superparticular divisions. (6x) Wilson's enharmonic, also on Hofmann's list of superparticular divisions.*

INDEX	NUMBERS							PYKNA
1x	16						15	16/15
2x	32			31			30	32/31 · 31/30
3x	48		47		46		45	24/23 · 46/45
4x	64	63		62	61		60	64/63 · 21/20
5x	80	79	78	77	76		75	20/19 · 76/75
6x	96	95	94	93	92	91	90	96/95 · 19/18

2-5. *Ptolemy's interpretation of Aristoxenos's genera.*

ENHARMONIC	
40/39 · 39/38 · 19/15	44 + 45 + 409
SOFT CHROMATIC	
30/29 · 29/28 · 56/45	59 + 60 + 379
HEMIOLIC CHROMATIC	
80/77 · 77/74 · 37/30	66 + 69 + 363
INTENSE CHROMATIC	
20/19 · 19/18 · 6/5	89 + 94 + 316
SOFT DIATONIC	
20/19 · 38/35 · 7/6	89 + 142 + 267
INTENSE DIATONIC	
20/19 · 19/17 · 17/15	89 + 192 + 217

sions of the pyknon while those of Ptolemy favor the 1:2 proportion, although in some instances the sub-intervals must be reordered so that the melodic proportions are the canonical order; small, medium and large. This principle was also enunciated by Aristoxenos, but violated by Archytas, Didymos, and Ptolemy himself in his diatonic tunings.

A more direct method of calculating the divisions is to use the following formulae (Winnington-Ingram 1932; Barbera 1978) where x/y is the interval to be linearly divided:

1/1 $2x/(x+y)\cdot(x+y)/2y=x/y$,
1/2 $3x/(2x+y)\cdot(2x+y)/3y=x/y$,
2/1 $3x/(x+2y)\cdot(x+2y)/3y=x/y$.

Finer divisions may be defined analogously; if a/b is the desired proportion and x/y the interval, then $(a+b)\cdot x/(bx+ay)\cdot(bx+ay)/(a+b)\cdot y=x/y$.

The final set of tetrachords given by Ptolemy are his interpretations of the genera of Aristoxenos (2-5). Unfortunately, he seems to have completely misunderstood Aristoxenos's geometric approach and translated his "parts" into aliquot parts of a string of 120 units. Two of the resulting tetrachords are identical to Eratosthenes's enharmonic and chromatic genera, but the others are rather far from Aristoxenos's intent. The Ptolemaic version of the hemiolic chromatic is actually a good approximation to Aristoxenos's soft chromatic. Aristoxenos's theories will be discussed in detail in chapter 3.

The late Roman writers

After Ptolemy's recension of classical tuning lore, a few minor writers such as Gaudentius (fourth century CE) continued to provide tuning information in numbers rather than the fractional tones of the Aristoxenian school. Gaudentius's diatonic has the familiar ditone or Pythagorean tuning, as does his intense chromatic (chroma syntonon), 256/243 · 2187/2048 · 32/27 (Barbera 1978).

The last classical scholar in the ancient arithmetic tradition was the philosopher Boethius (sixth century CE) who added some novel tetrachords and also hopelessly muddled the nomenclature of the modes for succeeding generations of Europeans. Boethius's tuning for the tetrachords in the three principal genera are below:

ENHARMONIC: 512/499 · 499/486 · 81/64
CHROMATIC: 256/243 · 81/76 · 19/16
DIATONIC: 256/243 · 9/8 · 9/8

These unusual tunings are best thought of as a simplification of the Pythagorean forms, as the limma (256/243) is the enharmonic pyknon and the lowest interval of both the chromatic and diatonic genera. The enharmonic uses the 1:1 division formula to divide the 256/243, and the 19/16 is virtually the same size as the Pythagorean minor third, 32/27.

The medieval Islamic theorists

With the exception of Byzantine writers such as Pachymeres, who for the most part repeated classical doctrines, the next group of creative authors are the medieval Islamic writers, Al-Farabi (950 CE), Ibn Sina or Avicenna (1037 CE) and Safiyu-d-Din (1276 CE). These theorists attempted to rationalize the very diverse musics of the Islamic cultural area within the Greek theoretical framework.

In addition to an extended Pythagorean cycle of seventeen tones, genera of divided fifths and a forty-fold division of the the string (Tanbur of Baghdad) in Al-Farabi, several new theoretical techniques are found. Al-Farabi analogizes from the 256/243 · 9/8 · 9/8 of the Pythagorean tuning and proposes reduplicated genera such as 49/48 · 8/7 · 8/7 and 27/25 · 10/9 · 10/9. Avicenna lists other *reduplicated* tetrachords with intervals of approximately 3/4 of a tone and smaller (see the Catalog for these genera). The resemblance of these to Ptolemy's equable diatonic seems more than fortuitous and further supports the notion that *three-quarter-tone* intervals were in actual use in Near Eastern music by Roman times (second century CE). These tetrachords may also bear a genetic relationship to neutral-third pentatonics and to Aristoxenos's hemiolic chromatic and soft diatonic genera as well as Ptolemy's intense chromatic.

Surprisingly, I have been unable to trace the apparently missing reduplicated genus, 11/10 · 11/10 · 400/363 (165 + 165 + 168 cents) that is a virtually equally-tempered division of the 4/3. Lou Harrison has pointed out that tetrachords such as this and the equable diatonic yield scales which approximate the 7-tone equal temperament, an idealization of tuning systems which are widely distributed in sub-Saharan Africa and Southeast Asia.

Other theoretical advances of the Islamic theorists include the use of various arrangements of the intervals of the tetrachords. Safiyu-d-Din listed all six permutations of the tetrachords in his compendious tables, although his work was probably based on Aristoxenos's discussion of the permutations of the tetrachords that occur in the different octave species.

At least for expository purposes, the Islamic theorists favored arrangements with the pyknon uppermost and with the whole tone, when present, at the bottom. This format may be related to the technique of measurement termed *messel,* from the Arabic *al-mithal,* in which the shorter of two string lengths is taken as the unit, yielding numbers in the reverse order of the Greek theorists (Apel 1955, 441–442.).

The so-called *neo-chromatic* tetrachord (Gevaert 1875) with the augmented second in the central position is quite prominent and is also found in some of the later Greek musical fragments and in Byzantine chant (Winnington-Ingram 1936) as the *palace mode*. It is found in the *Hungarian minor* and *Gypsy* scales, but, alas, it has become a common musical cliché, the "snake-charmer's scale" of the background music for exotic Oriental settings on television and in the movies.

The present

After the medieval Islamic writers, there are relatively few theorists expressing any great interest in tetrachords until the nineteenth and twentieth centuries. Notable among the persons attracted to this branch of music theory were Helmholtz ([1877] 1954) and Vogel (1963, 1967, 1975) in Germany; A. J. Ellis (1885), Wilfrid Perrett (1926, 1928, 1931, 1934), R. P. Winnington-Ingram (1928, 1932) and Kathleen Schlesinger (1933) in Britain; Thorvald Kornerup (1934) in Denmark; and Harry Partch (1949) and Ervin Wilson in the United States. The contributions of these scholars and discoverers are listed in the Catalog along with those of many other workers in the arithmetic tradition.

After two and a half millennia, the fascination of the tetrachord has still not vanished. Chapter 4 will deal with the extension of arithmetical techniques to the problem of creating or discovering new tetrachordal genera.