Optimal Tuning Systems

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Introduction

Historically, scales and tuning systems throughout the world have evolved according to number of constraints, including:

1) a *fixed set* of possible pitches
2) a set of *"ideal" tunings* for certain intervals
3) a set of cultural and/or theoretical weights reflecting the importance of certain *intervals*, and "keys"
4) a value (e.g. the octave) around which scales *repeat*
5) the ability to begin "scales" in the system on *different pitches*

This paper describes a mathematical formulation of this idea of a tuning system, and a corresponding algorithm for optimizing tuning systems according to these criteria. Implicit in the notion of constraints and optimization is a distance function or metric according to which we can judge a "best" choice. We will present a formulation in which we are able to obtain a unique solution to this problem ("deterministic" in the sense of obtained as a solution to a particular set of equations), and then demonstrate the derived tuning system using a real-time public domain software application that we have developed for this purpose.[1]In addition, we will present several examples of usual and unusual optimal tunings, and compare them mathematically and musically to well-known historical models. Finally, we will discuss ramifications of this technique for music theory, composition, and ethnomusicology.

To give our work some context, we proceed from the thesis that it is possible to take the view that western music tunings, and the evolution from just intonation, via mean-tone, through well-temperament, and finally to 12-tone equal temperament may be seen as a gradual and continuous attempt at resolving an embodiment of the

[1] This software, and other materials relating to this research can be found at http://music.dartmouth.edu/~larry/owt

constraints listed above. In a sense, well-temperaments remain the most sophisticated solutions to the "tuning problem," since they attempt a resolution in the most complete way. They are also the most difficult tuning systems to design, and perhaps as a consequence, had a comparatively short historical life.

By examining the idea of well-temperament in a general and mathematical way, we suggest a well-defined approach to achieving the *optimal resolution of a system's constraints*. Informally, this is a way to make the "best" possible tuning system, given some well-defined ideas of what "best" might mean. Among other things, we are interested in describing a method for the development of classes of scales and tunings of use to composers and scholars in many ways (including computer-based adaptive tunings, and the analysis of tuning systems around the world).

Well Temperament and the "Historical Tuning Problem"

Almost all tuning systems (even those not based on simple primes and rational ratios), whether explicitly or implicitly, need to accommodate a basic number theoretic fact: given two distinct primes p, q, there are no positive integers m, n, such that $p^m = q^n$. This implies that *any* finite tuning system containing more than one prime will at some point be "out of tune" with itself. The most well-known example of the problem is the "wolf-fifth," central to the development of tuning systems.

In this sense, all tuning systems are compromises (e.g., mean-tones, well-temperaments, and finally, 12-ET). Our framework suggests a formal and mathematical approach to finding a compromise: *optimization*. By stipulating a simple set of formal criteria (see **Figure 1**), we develop a formal theory for tuning system construction.

Tuning Matrices and Error Functions

One way to view a tuning system is as a half-matrix of relationships. More precisely, for *n* pitches, there are $(n^2 - n)/2$ intervals. The size of an interval may be written in the appropriate position in a pairwise-relationship matrix. A 12-note scale has 12 minor 2nds, 11 major 2nds, and so on. For example, consider the well-known well-temperament Werckmeister 3[2] (see **Figure 2**). From its matrix we can compute an *error matrix*, comprised of the distances between *actual* and *ideal* intervals (see **Figure 3).**

The error matrix in **Figure 3** quantifies W3's deviation from its ideal tunings. The sum of the entries in the matrix (after appropriate weighting) is the *error function* — a measure of how well a tuning fits its own "design" criteria. By *minimizing* that error function, we *optimize* the error matrix. This suggests a procedure for evaluating and creating tuning systems parametrically, according to explicit criteria. A tuning system is thus described by and created from its higher-level features rather than its actual pitches (see **Figure 4).**

We describe a general mathematical solution for minimizing the error function, based on a least squares approach. The advantage of least squares is that it admits an analytic solution (other optimality criteria may require the use of non-deterministic approaches, such as genetic algorithms, which we have also implemented). We see this as a general result in the field of tuning theory, providing a kind of "theory of all tuning systems," for a fairly general definition of "tuning system."

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[2] Rasch, Rudolf, editor. *Andreas Werckmeister, Musicalische Temperatur*, 1691, Utrecht, The Diapason Press, 1983

FIGURES

Figure 1: Criteria for Tuning System Optimization

- *repeat factor* (e.g., an octave)
- set of "desired" or *ideal ratios* $\mathbf{I_1, \ldots I_n}$ (= (number of pitches) -1) for intervals (that is, all intervals between scale pitches)
- set of relative weights for these intervals $\mathbf{i_1, \ldots i_n}$
- set of relative weights for "keys," $\mathbf{k_1, \ldots k_n}$ or the fixed pitches in the scale to which intervals are measured
- fixed *number of pitches* $\mathbf{n + 1}$ (e.g., a scale, which has one more pitches than the number of intervals)

Figure 2: W3 as a ½ matrix

	C	C#	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C		90	192	294	390	498	588	696	792	888	996	1092	1200
C#			102	204	300	408	498	606	702	798	906	1002	1110
D				102	198	306	396	504	600	696	804	900	1008
Eb					96	204	294	402	498	594	702	798	906
E						108	198	306	402	498	606	702	810
F							90	198	294	390	498	594	702
F#								108	204	300	408	504	612
G									96	192	300	396	504
G#										96	204	300	408
A											108	204	312
Bb												96	204
B													108

Each diagonal represents one interval.

Figure 3: Error Matrix (four intervals only) for W3

	C	C#	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C		–	12	–	6	0	–	6	–	–	–	–	1200
C#			–	0	–	24	0	–	0	–	–	–	–
D				–	6	–	12	6	–	6	–	–	–
Eb					–	0	–	18	0	–	0	–	–
E						–	6	–	18	0	–	0	–
F							–	6	–	6	0	–	0
F#								–	0	–	24	6	–
G									–	12	–	12	6
G#										–	0	–	24
A											–	0	–
Bb												–	0
B													–

Only M2nds, M3rds, P4ths, and P5ths are shown. Column entries are the absolute error (in cents) for each occurrence of those intervals (where the ideal intervals, respectively, are 9/8, 5/4, 4/3, and 3/2).

Figure 4: W3 derived from a few just intervals

	C	C#	D	Eb	E	F	F#	G	Ab	A	Bb	B
Ideal Interval (in cents)	–	100	204	300	386	498	600	702	800	900	1000	1100
Interval Weight	–	1	1	1	101	201	1	1201	1	31	1	1
Key Weight	1	61	11	1	1	801	1	1	1	101	11	1

As an example of using our framework to derive a historically well-known scale, we used a search-space algorithm to generate W3 using a small subset of just intervals (9/8, 5/4, 4/3, 3/2), filling in the gaps with 12-ET intervals. The search algorithm, in conjunction with our optimization technique, found the interval and key weight sets above. These, along with the specified ideal interval set, "describe" W3. Note that the key and interval weights correspond closely to W3's actual structure. For example, the highest weighted intervals found are for the M3, P4, and P5. In addition, according to at least one measure, that of triadic intonations (see Rasch, cited in our proposal), F major is the "best" key in W3. (Note also that this is just one of many such weighting sets that will achieve the same result for a specific set of ideal intervals).