

A Mathematical Model for Optimal Tuning Systems

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1. Introduction

In this paper we propose a mathematical framework for the optimization of tuning systems. We begin with an informal definition of “tuning system.” We then propose five general constraints that seem common to their evolution. The central idea is a formulation of those constraints as a set of numerical preferences and weights. From these parameter values we use appropriate optimization methods to produce an optimal tuning for a specific set of values. Finally, we consider some historical tunings from this perspective, and use the framework to generate a few examples of novel tuning systems.

2. Tuning systems

A tuning system is a set of intonations for intervals or pitch classes. A tuning system might be used by a musical culture, group of musicians, or even a single composer. Such a system may also serve as an abstraction, or model, for the derivation of any number of related systems and sub-systems.

Smaller, functional subsets of pitches, such as scales, are extracted from a larger tuning system for specific musical purposes. Examples include the formation of major and minor (and other) scales from 12-tone equal temperament (12-ET), or the various Javanese pathet (*manyura*, *nem*, *sanga*, etc.) which are subsets of *slendro* and *pélog* tuning [Perlman pp. 40-43].

Tuning systems are neither static nor rigid. Although most musical cultures need some agreed-upon standard for musicians to tune their instruments and sing to, tuning systems evolve and fluctuate over time and in space (i.e., historically and geographically) and

vary stylistically within musical practice. Most musical cultures have some standard or canonical tuning, articulated in either oral or in written traditions. Such a system may often be canonized in a specific instrument, like the piano, or the *gendér* in Central Javanese music which may hold the tuning for an entire gamelan.

We are interested here in a formal framework for tuning systems themselves, not the intricate (but no less important) musical variations and manifestations of such a system. Musicians deviate freely and artistically from standardized tunings in fluid, complex ways. For example, the many musical genres that share the nomenclature and intervallic template of 12-ET (like jazz and blues) are intonationally diverse. But the complexities and nuances of intonational usage associated with established tuning systems is beyond the scope of this paper.

Culturally- and historically-specific constraints may influence the formation of tuning systems. A new system that resembles a pre-existing one is often desirable, as in Central Javanese gamelan tunings which reference well-known gamelan.⁵ A tuning system might adapt over time in the performance of an evolving body of music. This latter consideration is an important factor in the historical evolution of tunings in European music over the past millennium, including just intonations [JIs], mean-tones, well-temperaments [WTs], equal temperaments [ETs], and 20th century experimental tuning systems.

From a formal, abstract perspective tuning systems can be seen as specific attempts to solve certain problems, and understood as the resolution of a particular set of intonational constraints. The genesis of these problems — whether they emanate from issues of culture, economy, convenience, aesthetics, or some complex combination of all of these— is another issue. Our focus is on a relatively small set of important issues common to the creation of tuning systems, whose natural quantitative formulation enables the use of *optimization* techniques for analysis. We believe this approach has important implications towards a deeper understanding of tuning and even musical style.

3. Five Constraints

Tuning systems through history and across cultures have used a set of complex compromises to account for some or all of the following constraints:

1. *Pitch set*: use of a fixed number of pitches (and consequently, a fixed number of intervals);
2. *Repeat factor*: use of a *modulus*, or *repeat factor*⁶ for scales, and for the tuning system itself (i.e., something like an octave);
3. *Intervals*: an idea or set of ideas of correct or ideal intervals, in terms of frequency relationships;
4. *Hierarchy*: a hierarchy of importance for the accuracy of intervals in the system;
5. *Key*: a higher-level hierarchy of the relative importance of the “in-tuneness” of specific scales or modes begun at various pitches in the system.

Most tuning systems attempt to resolve some or all of the five constraints listed above. The best fit may also reflect and incorporate other theoretical, cultural, historical, and even aesthetic factors. These five constraints, which appear to operate at a different, less culturally specific level, can be stated formally and mathematically. In this formulation, there is an associated *weighting system* that allows the generation of any tuning system with a fixed number of pitches, repeat-factor, and some set of ideal intervals. Not all tuning systems, of course, consider all of these constraints in their construction. Neither are they exhaustive.⁷ However, these constraints constitute an economical and musically reasonable set capable of providing an interesting analysis of any tuning system.

4. Comments on the five constraints

Some concept of a *number of pitches* seems to be nearly universal, primarily, perhaps, for practical reasons. In actual performance practice the situation is usually more complex. Style-specific ideas of “between-ness” and intonational variation are sometimes well-articulated in theory and pedagogy, yet it is difficult to completely account for intra- and inter-cultural variation. For that reason, most theoretical tuning literature has focused on

canonical tuning systems. We adopt a similar approach here, assuming a fixed number of “discrete” [Burns and Ward p. 243] pitches for a tuning system.

While musics vary widely in the numbers of named pitches in use, having a fixed number of pitches offers practical and cognitive advantages. The most obvious benefit is economical, as it affects the construction of instruments. There are pedagogical and musical advantages to a finite, even moderately sized set of pitches from which to learn to compose, improvise, sing, and perform common repertoires [Dowling and Harwood p. 92; Burns and Ward p. 244]. Some authors [i.e., Lerdahl] have proposed cognitive explanations for certain numbers of pitches in a system. Whatever the specific optimal numbers may be, cultures seem to eventually agree on a specific number of pitches for a tuning system.⁸

Repetition of an interval set at some fixed interval, or *modulus*, is common, if not universal, and is related to *having* a fixed set of pitches. Both music theory and cognition often distinguish between *pitch height* and *pitch chroma* [Shepard 1964; 1982], the latter usually referring to pitches whose frequencies are related as powers of two, or octaves. Although the modulus interval is usually the octave (or something close to it), this is not always the case, as in the well-known Pierce-Boehlen scale [Matthews and Pierce; see also, Moreno], or the stretched octaves of Central Javanese tuning [Surjodiningrat et al.; Sethares; Polansky 1984]. The universality of the octave is not as much in dispute. However, the extent to which its prevalence is best accounted for by hard-wired or learned cognitive mechanisms is uncertain [e.g., Burns and Ward 262-264]. Regardless, most tuning systems seem to *repeat*, or “cycle,” [Dowling and Harwood 19] at some fixed interval. This interval, in practice, can be somewhat flexible, incorporating spectral considerations (such as the stretched octaves of piano tuning), musical factors, or some combination of the two. Given the utility of a fixed set of pitches, a reasonably constant repeat-factor is a likely corollary.⁹ In our framework we do not assume a specific interval of repetition — only that the system repeats at a reasonably constant interval.¹⁰

Tuning systems that place a greater importance on some intervals than others are called hierarchical [Krumhansl, 1987a, 1990, in particular, Chapter 10]. The mathematical simplicity of an interval is often correlated with its importance (such as a “pure” 5th, octave, or 3rd). There are cognitive and acoustical arguments for the importance of these intervals—they are easily heard, measured, and produced, and are likely candidates to function as generators for larger sets of intervals. As a consequence, the mistuning of these important intervals may be of greater concern. Even in 12-ET, this hierarchy exists: the octave (2/1) is exact, the 5th is extremely close (2¢ difference to 3/2), and the 3rd (14¢ difference to 5/4) is furthest from a simple, rational interval.

Subsets of tuning systems (scales, modes, *pathet*, or *rags* [Jairazbhoy], etc.) may similarly demonstrate such a hierarchy. Many, if not most, bluegrass songs, for example, seem to be in open keys on the guitar, mandolin, and fiddle (G, D, A, E), and it is common for bluegrass guitarists to tune to an “open G chord,” privileging its tuning slightly over 12-ET. In other words, it is more important that the key of G be “in tune” than the key of C#. Historically, European art music has implemented these kinds of relationships in mean-tone tunings and well-temperaments. Key hierarchies also occur in the *pathet* (roughly: *scales*) of Central Javanese gamelan. The tuning system called *slendro* has three *pathet* (*manyura*, *nem*, *sanga*), which have complex usage relationships implying intonational constraints for particular melodic patterns and intervals. [Perlman pp. 44-45; Sumarsam pp. 142-143]

It is important to state *a priori* that the mathematical formulation outlined here does not assume that tuning systems are based on small, rational intervals. Indeed, many tuning systems (such as *slendro* and 12-ET) deviate significantly from those kinds of intervals. Nor do we mean to engage longstanding discussions of “categorical perception” [Burns and Ward pp. 250–254; Sethares p. 50] or “consonance” [Tenney 1988]. The purpose of this framework is to consider tuning systems through the analysis of stylistically non-specific variables such as number of pitches, repeat factor, ideal intervals, interval and key weights.

5. Historical Example: Well-temperament

Examination of a few well-known constructs from historical European tuning theory can illustrate some of the motivations for the mathematical framework described below.

Several concepts central to our framework are introduced here: the *interval matrix*, *ideal intervals*, the associated *error matrix*, and the *error function*.

The example below shows a standard Just Diatonic scale:

1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	(ratios)
C	D	E	F	G	A	B	C	(note names)
0¢	204¢	386¢	498¢	702¢	884¢	1088¢	1200¢	(cents values of intervals)

By definition, JIs are “in tune” in one key (possibly a few) and problematic in others. Certain intervals that represent differences between a fixed number of scale degrees will differ from some *ideal interval* which appears elsewhere in the scale. In the Just Diatonic scale, the most important such interval (aside from the 2/1 octave) is the 3/2 P5th. The musical 5th between the 2nd and 6th degrees of the scale above (40/27, the well known “wolf-fifth”) differs from that between the 1st and 5th or the 3rd and 7th scale degrees.¹¹ Central to this phenomenon is that intervals in this scale are built on more than one prime: in this case, the primes 2, 3 and 5. A tuning system such as JI can’t have pure intervals involving different primes and still be in tune with itself. The wolf-fifth is only one example, if perhaps the most famous.¹² An even simpler version of this phenomenon can be seen in the well-known Pythagorean comma,¹³ which involves only the primes 2 and 3.

This fundamental problem, which be called the *collision of primes*, has no simple resolution, and has motivated the development of many tuning systems. It is also, in part, a consequence of the constraints of a repeating tuning with a fixed number of notes. WTs, and ultimately, equal-temperaments (ETs), approximate ideal intervals in more complex ways, moving towards the equality of derived scales beginning on different notes.

We might refer to this as the *historical tuning problem*,¹⁴ a musical/mathematical ramification of the insolvability of the Diophantine equation $p^n \neq q^m$ for distinct primes p and q .¹⁵ The problem for music theory is quite general, and might more appropriately be referred to not by its species name (“wolf”), but by its family name: the “canidae interval,” a more general description of any interval that is the result of a collision of primes and therefore necessitates some sort of compromise in the tuning system itself¹⁶.

But this “collision” will occur in any system with irregular intervals — not just rational intervals. Thus, any system (except for ET) that has some set of ideal intervals, fixed set of pitches, and repeat factor needs to develop systematic compromises in order to resolve this problem.

The *interval matrix* is a useful way to encode a tuning system. The i,j (row, column) entry in this matrix is the interval between notes i and j . Figure 1 is an example of the $\frac{1}{2}$ -matrix¹⁷ of the JI diatonic scale¹⁸ clearly showing the wolf 5th:

	C	D	E	F	G	A	B	C
	(1/1)	(9/8)	(5/4)	(4/3)	(3/2)	(5/3)	(15/8)	(2/1)
C		9/8	5/4	4/3	3/2	5/3	15/8	2/1
D			10/9	32/27	4/3	40/27	5/3	16/9
E				16/15	6/5	4/3	3/2	8/5
F					9/8	5/4	45/32	3/2
G						10/9	5/4	4/3
A							9/8	6/5
B								16/15

Figure 1: $\frac{1}{2}$ Interval Matrix of Just Diatonic Scale. Intervals are in ratios within one octave. 5ths are in bold. The wolf (40/27) is the only non-ideal P5th. Note that all M3rds (C-E, F-A, G-B) are ideal intervals of 5/4.

An *ideal tuning* would be one in which the i,j entry only depends on $|i-j|$ — each entry is equal to an ideal interval. In the *ideal interval matrix*, values on the diagonals are constant and equal to the ideal ratio. The ideal interval matrix is equivalent to the interval matrix only in ET. Any unequal interval propagates itself through the matrix, causing irregularity.

The *error matrix* of a tuning system contains the differences between the entries of the interval matrix and the respective entries in the *ideal* interval matrix. Figures 2 and 3 show the interval matrix and a portion of the error matrix for one of the most studied WTs in history: the so-called “Werckmeister III (W3)” tuning, devised by Andreas Werckmeister, an influential 17th-century music theorist and keyboard tuner.¹⁹

	C	C#	D	E^b	E	F	F#	G	A^b	A	B^b	B	C
C		90	192	294	390	498	588	696	792	888	996	1092	1200
C#			102	204	300	408	498	606	702	798	906	1002	1110
D				102	198	306	396	504	600	696	804	900	1008
E^b					96	204	294	402	498	594	702	798	906
E						108	198	306	402	498	606	702	810
F							90	198	294	390	498	594	702
F#								108	204	300	408	504	612
G									96	192	300	396	504
G#										96	204	300	408
A											108	204	312
B^b												96	204
B													108

Figure 2: W3 ½-matrix. Each diagonal is a specific interval. “Keys” correspond to rows. All keys in W3 are considered reasonably good. Values are in cents.

	C	C#	D	E^b	E	F	F#	G	A^b	A	B^b	B	C
C	—	—	—	4	0	—	6	—	—	—	—	—	—
C#		—	—	—	22	0	—	0	—	—	—	—	—
D			—	—	—	10	6	—	6	—	—	—	—
E^b				—	—	—	16	0	—	0	—	—	—
E					—	—	—	16	0	—	0	—	—
F						—	—	—	4	0	—	0	—
F#							—	—	—	22	6	—	—
G								—	—	—	10	6	—
G#									—	—	—	22	—
A										—	—	—	—
B^b											—	—	—
B												—	—

Figure 3: W3 error ½-matrix. Three (diagonal) intervals (M3rd, P4th, P5th) are shown. Note that the “central keys” (C, F) have smaller errors for the 3rd and 5th (another central key, G, is the inversion of F, and is not shown in the ½-matrix). The supertonic key (D) is also relatively good, since it is a likely next key in tonal music after the tonic and dominant.

ideal interval matrix is not shown, but ideal values for the P5th, M3rd, and P4th, are 702, 486, and 698 respectively (note that these do not occur in the key of C in W3).²⁰

A comparison of these ½-matrices shows that in a WT system like W3 the degree of error varies considerably over intervals and keys (shown here as diagonals and rows). Smaller errors in W3 tend to be found in central keys (beginning on the first, 5th, and 7th degrees) and in important intervals like the M3rd and P5th.

All WTs (and W3 in particular) illustrate the technique we describe below. They are also an excellent, clearly articulated set of examples that tuning systems seem to evolve as attempts to reconcile ideas of key, ideal interval, a fixed number of pitches, and something like the octave into an *optimal* tuning system. Well-documented by the 18th century theorists who developed them, WTs have been discussed and analyzed a great deal ever since [Jorgensen, Lindley, Barbour, Donahue].

6. Mathematical Formulation

A rigorous notion of an *optimal tuning system* may be formulated by providing a formal, mathematical framework for some seemingly universal and important aspects of tuning. We begin by formalizing the five constraints on tuning systems described above. Next, we define an abstract representation of the *interval matrix* and the associated *error matrix* as well as an *error function* that uses the error matrix as input. We then minimize this error function using least squares to find the optimal tuning.

The five constraints on tuning systems are formalized as follows:

1. Pitch set: let a_1 to a_n be a set of n pitches, none equal to 0.
2. Repeat factor: let $\omega > a_n$ be the repeat factor of the tuning system.
3. Intervals: let I_1 to I_n represent the ideal intervals.
4. Hierarchy: let i_1 to i_n be *interval weights* to represent the desired accuracy of the n intervals in the tuning system.

5. Key: let k_0 to k_n be *key weights* to represent the fixed pitches in the tuning system to which intervals are measured.

Using this notation, the interval matrix M for a set of n pitches, a_1 to a_n , is written as:

$$M = \begin{pmatrix} 0 & a_1 & \dots & a_{n-1} & a_n \\ \omega - a_1 & 0 & \dots & a_{n-1} - a_1 & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega - a_{n-1} & \omega + a_1 - a_{n-1} & \dots & 0 & a_n - a_{n-1} \\ \omega - a_n & \omega + a_1 - a_n & \dots & \omega + a_{n-1} - a_n & 0 \end{pmatrix}.$$

The matrix M has $n+1$ rows and $n+1$ columns with zeros along the diagonal. The entry in row i and column j is $a_j - a_i$ if $i \leq j$, or $\omega + a_j - a_i$ if $i > j$. The matrix M specifies all possible intervals of the tuning system. All instances of a particular interval are found on the diagonals of the matrix.

The ideal interval matrix L represents the desired interval for each entry in the matrix M :

$$L = \begin{pmatrix} 0 & I_1 & \dots & I_{n-1} & I_n \\ I_n & 0 & \dots & I_{n-2} & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_2 & I_3 & \dots & 0 & I_1 \\ I_1 & I_2 & \dots & I_n & 0 \end{pmatrix}.$$

Note that L is circulant and is generated by rotating each row to the right relative to the previous row.

The error matrix is the difference between the interval matrix M and the ideal interval matrix L . The total error of a tuning system is defined as a function of the error matrix. Natural choices for this function include the sum of the absolute values of the entries, known as the L^1 error, or the square root of the sum of the squares of the entries, known as the L^2 error.²¹ We use the L^2 error, to simplify differentiation for a least-squares solution. We have also developed, however, a genetic-algorithm solution that allows other error functions to be used.²²

In the absence of any key or interval hierarchy (i.e., the interval weights and key weights are all equal) the error function is:

$$E(\vec{a}) = \sum (M - L)^2,$$

where the vector \vec{a} contains the n pitches, a_1 to a_n , and the exponentiation and summation are applied element-wise to the matrix. In Appendix A, we prove that ET is always the optimal solution for this case, independent of the ideal intervals specified in the matrix L . This result is in accord with historical evolution of tuning systems: if no key or interval is preferred over any other key or interval, ET is the best solution, since by definition it has no intonational differences between keys. If, however, some keys or intervals are more important than others, as in WTs, then ET may not be the best solution in the least-squares sense.

To formalize the notion of the relative importance of intervals and keys, we use interval and key weights. These weights can be applied to the error function through a weight matrix W :

$$W = I * K$$

where the ‘*’ operator denotes a Hadamard, or element-wise product, and the interval-weight matrix I and key-weight matrix K are defined as:

$$I = \begin{pmatrix} 0 & i_1 & \dots & i_{n-1} & i_n \\ i_n & 0 & \dots & i_{n-2} & i_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ i_2 & i_3 & \dots & 0 & i_1 \\ i_1 & i_2 & \dots & i_n & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_0 & k_0 & \dots & k_0 & k_0 \\ k_1 & k_1 & \dots & k_1 & k_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-1} & k_{n-1} & \dots & k_{n-1} & k_{n-1} \\ k_n & k_n & \dots & k_n & k_n \end{pmatrix}.$$

The weighted version of the error function is therefore

$$\hat{E}(\vec{a}) = \sum W * (M - L)^2$$

in matrix notation and

$$\hat{E}(\vec{a}) = \sum_{i=0}^n k_i \left[\sum_{j=0}^{i-1} i_{n+j-i+1} (\omega + a_j - a_i - I_{n+j-i+1})^2 + \sum_{j=i+1}^n i_{j-i} (a_j - a_i - I_{j-i})^2 \right]$$

in double-summation notation.

For each set of constraints specified in the matrices I , K and L , there is a unique solution that minimizes the weighted error function. This solution is called the *optimal tuning system*, and it is a set of pitches a_1 to a_n . While the optimal tuning is unique to a given set of constraints, the converse is not true: there is not necessarily a unique set of constraints that will generate a given tuning. In other words, multiple sets of constraints can generate the same tuning, within a specified tolerance.²³

For example, we can approximate W3 within an average error of 0.5 cents using one (Pythagorean) ideal interval and the following interval and key weights:

Ideal intervals	90	204	294	408	498	612	702	792	906	996	1110	
Int. weights	-	-	-	-	-	-	1	-	-	-	-	
Key weights	-	0.3	-	0.15	0.05	0.05	0.1	-	0.2	0.05	0.1	-

Though many sets of constraints could generate W3, these particular constraints illustrate an interesting alternative view of the tuning: W3 can be generated by specifying good P5s in the *outer keys*.²⁴ This is, perhaps, surprising in light of common formulations of WTs in which “inner” keys are considered to be central to their design. In other words, the idea of tuning to central keys may be historically true but not mathematically necessary.²⁵

7. Comments

Using this framework, we can design and produce tunings incorporating historical and cross-cultural criteria. With some high degree of accuracy, we can approximate criteria for the development of historical tuning systems. By replicating a system heuristically, we might be able to infer something about the criteria important to its design, such as the key and interval weights that would produce it from a given set of ideal intervals (or vice versa). This approach can also be used to invent new systems.

In all cases, the tuning system is specified parametrically. This framework considers tuning systems as a formal description of their higher level “features” (the five constraints) rather than by their “note-by-note” intervals. A given set of features uniquely describes one tuning system. This can be an interesting and powerful approach to the

study of scales, tuning systems, and tuning in general, as well as a creative tool for new compositional ideas.

8. Examples: Optimal Well Temperaments

A convenient point of comparison for our model is W3, since it is an historical version of the method proposed here. "... [W3] was the first unequal temperament to allow satisfactory performance of all possible tonalities" [Ledbetter p. 38]. Using the information contained in the interval and error matrix, we can rigorously analyze and expand upon statements such as the above description of W3, as well as other tunings. In some sense, all tuning systems other than ETs are well-tempered: they must attempt to resolve some or all of these constraints. From this perspective, our framework might be alternatively described as a way of creating *optimal well-temperaments* (OWTs).

Example 1: OWT1 and OWT2 and minimal mean-tempering

As a way of exploring the possibilities of our framework to formally generalize historical criteria for tuning systems, we created several new WTs. Rasch, Chalmers [1974], and others have proposed simple, reasonable measurements of tuning systems.²⁶ Rasch, in his consideration of Werckmeister's tunings, measures the *mean-tempering* of "all consonant intervals, which is equal to the mean tempering of all triads, or of all keys." [Rasch pp. 38–46]. This is the mean difference of the three intervals in the major chord tuned as 1/1, 5/4, 3/2 (ideal intervals) and the actual intervals in the tuning over 12 keys. Rasch's measure is thus an *error function*, distinct from but related to the one used in this paper, for a tuning system given a set of ideal intervals. Rasch uses this function to measure the degree to which Werckmeister's tuning systems are in tune. This gives some idea of the in tune-ness of a temperament vis-a-vis the pure M3rd and P5th (and consequently, major triads). In addition, Rasch's measure provides us with a meaningful way to measure the results of some simple experiments in generating new WTs.

W3 is exemplary in its mean-tempering of 10.43¢, which is the same as 12-ET (and can be shown to be an absolute minima²⁷). An historical WT often known as Young 2

(sometimes considered to be an improvement on W3 [Jorgensen; Donahue]), also achieves this minimality.

Using our framework, we generated two new optimal tunings with the same minimal mean-tempering as W3, Young 2, and 12-ET. Using reasonable sets of ideal intervals, we found sets of weights²⁸ for two new optimal tunings (OWT1, OWT2). These two new tunings are maximally in tune by a specific measure: mean-tempering of triads. They have a great deal in common (ideal intervals, key and interval weights), theoretically and musically, with historical WTs. Yet, their musical implications and structure differ in important ways from their historical models.

<i>W3</i>	0	90.2	192.2	294.1	390.2	498.1	588.3	696.1	792.2	888.3	996.1	1092.2
<i>Young 2</i>	0	90.2	196.1	294.1	392.2	498.0	588.3	698.0	792.2	894.1	996.1	1090.2
<i>OWT1</i>	0	102.0	203.8	297.2	396.3	498.1	600.0	702.0	803.8	897.2	996.3	1098.1
<i>OWT2</i>	0	93.1	203.1	296.3	397.4	498.5	591.7	701.6	794.8	903.4	997.4	1091.4

Table 4: Four different minimally-tempered WTs

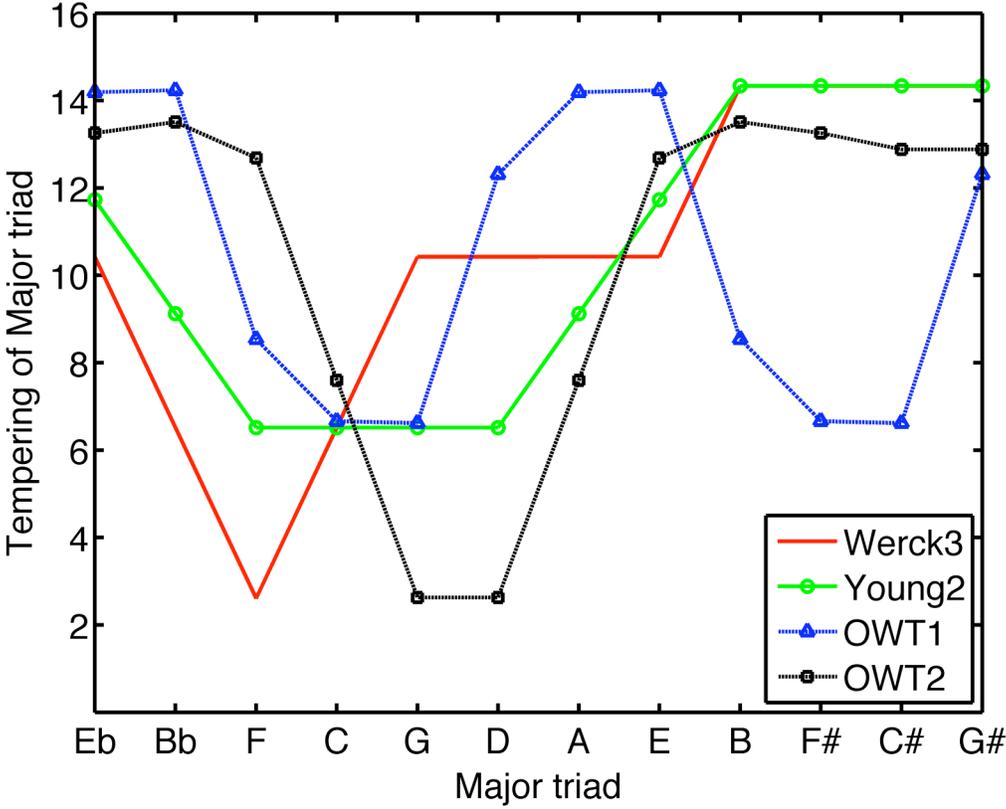


Figure 4: Tempering of triads in four minimally-tempered WTs.

The tempering of major triads in W3, Young 2, and the two generated tunings OWT1 and OWT2, is shown in Figure 4. The tunings can be viewed as four different distributions of the minimal error, each with different characteristics. Both W3 and OWT2 have certain triads that are more in tune than any in Young 2 or OWT1. OWT2 contains two adjacent triads with equal minima, while W3 which has only one minimal triad. This means that where W3 has one best key, OWT2 has two. OWT1 has two non-contiguous minima areas. For both OWT1 and OWT2 having several minima means that some other triads will be less in tune. This is historically unusual, and suggests, perhaps, a more chromatic musical style. Young 2 never achieves the triadic minima of W3 or OWT2, but has a wide adjacent region of relatively in-tune triads. It is important to reiterate that OWT1, OWT2, W3 and Young 2 all have the same minimum total mean-tempering error.

Our framework can generate new tunings that belong to a kind of *minimality class* (along with W3 and Young 2) but, at the same time, explore different musical possibilities. Furthermore, with our optimality method, we can represent those tunings parametrically as sets discreet features (weights, ideal intervals, etc.), corresponding to the constraints that shape the development of historical WTs.

Example 2: Septimal OWTs

New and experimental optimal tunings might even be created using our framework, for primarily speculative reasons. What if Werckmeister had listened to Bach in the morning and Lightnin’ Hopkins at night, and become fascinated with the flat m3rds and m7ths resulting from the use of the septimal intervals 7/6 and 7/4 respectively? These intervals, characteristic of blues and jazz, are prevalent in musical cultures around the world. Using our framework we can experiment, somewhat fancifully, with the creation of new tunings, adding septimal m3rds and m7ths to the ideal interval set while keeping M3rds and P5ths pure.²⁹

Septimal OWT1

Ideal intervals	100	204	267	386	498	600	702	800	900	969	1100
Int. weights	.0001	0.001	0.3	1	1	0.001	1	.001	0.001	1	0.001
Key weights	1	0.001	0.001	0.3	0.001	1	0.001	1	0.001	0.001	0.001

Septimal OWT2

Ideal intervals	100	204	267	386	498	600	702	800	900	969	1100
Int. weights	0.1	0.001	0.3	1	1	0.001	1	.001	0.001	1	0.001
Key Weights	1	0.001	0.001	0.001	0.001	1	0.001	1	0.001	0.001	0.001

The optimal tunings produced by these criteria are (in cents):

- Septimal OWT1: 0 42 206 272 386 491 543 704 764 877 977 1090 1200
- Septimal OWT2: 0 93 209 262 386 490 588 707 769 876 978 1093 1200

We recorded several Preludes and Fugues from the *Well Tempered Clavier* in these new septimal tunings, and in OWT1 and OWT2 (along with, for comparison, the conventional W3 and Young 2) [Polansky 2007].

Many of the intervals in these septimal OWTs are unusual, such as the $\frac{1}{4}$ -tone $m2^{\text{nd}}$ s, flat $m7^{\text{th}}$ s and $m3^{\text{rd}}$ s, but the usual 5^{th} s and 3^{rd} s are fairly well maintained. The resulting tunings, heard in the context of such a familiar musical work, might seem a little strange at first. Yet these tunings are, in some sense, a simple, transparent extension of the fundamental premises of WT. They give us some idea of how this framework generates new scales according to specific, demonstrable criteria (constraints). We like to think that both Bach and Werckmeister might have enjoyed these excursions into new tonalities, and perhaps even appreciated and understood their underlying ideas, enjoyed their harmonic excursions.

Conclusion

What are the implications of this framework for the consideration of the notion of a *tuning system* itself? Clearly, there are many issues involved in the development of a tuning system besides the specific criteria used here. A mathematical model such as ours does not account for cultural, aesthetic, historical, economic, or any number of intangible factors that might play a role in the development of a musical tradition.

Nevertheless, it might be proposed that *all* tuning systems, except for those that use rational intervals exclusively, are in fact, tempered. If that is at least partly true, then the method described here offers more than a historical analysis of the work of Werckmeister and his colleagues. The framework becomes, by extension, a general theory of tuning systems.

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Software: MatLab and Java tools for this framework are available at:

<http://eamusic.dartmouth.edu/~larry/owt/index.html>

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⁵ For example, Jody Diamond [personal communication] points out that the RRI [radio station] gamelan in Surakarta, Java, is widely imitated because of the many broadcasts and recordings made on this set of instruments.

⁶ The theorist Erv Wilson refers to this as the "interval of equivalence" [Burt p. 40].

⁷ For other examples of criteria sets for scales, tunings, and "musical pitch systems," see Erlich; Krumhansl [1987b]; Burns and Ward [p. 264]; Shepard [1987a]; Dowling and Harwood [pp. 90–100]. The last set of authors propose a set of "cognitive constraints on scale construction that seem to operate through much of the world" which differs from our framework in its focus on *scale*, rather than *tuning system* construction. As such, it takes into account factors — like Balzano *coherence* (or Rothenberg *propriety*), the number of intervals, and the idea of a small, modular interval — less pertinent to what Dowling and Harwood call "tonal materials" (our "tuning system" — their use of the term "tuning system" has a distinct meaning). However, our idea of key and interval weights is reflected in their "structural hierarchy" of a "modal scale."

Yasser and others (e.g. Kraehenbuehl and Schmidt) have postulated developmental mechanisms for tuning system development. There are, of course, many different and interesting examples of composers and theorists creating their own *best* tunings, all of which in a sense, attempt something like what we suggest here: optimization given (what are often artistic) constraints. Partch's well-known *monophonic scale* is one such example, as are the approaches of Blackwood, Lindley and Turner-Smith, and the recent work of Gräf (following Barlow) in his method of *scale rationalization* [also, see Carlos].

⁸ Regardless, many contemporary composers and theorists, such as Partch, Johnston, Sims, Tenney [e.g. 1987] and Fokker have developed tuning systems for their own use, with varying numbers of pitches per octave [see Keislar for an excellent survey]. A number of contemporary composers working in experimental intonation have used tuning

systems with an indefinite, dynamic or adaptive number of pitches per octave (for example, see [Polansky 1987]), but this approach is less common when the tuning system serves as some kind of standard for a musical community.

⁹ Burt surveys systems which repeat at values other than the octave, and describes some associated compositional experiments.

¹⁰ In fact, with a very simple alteration, the mathematics described later in this paper can eschew the assumption of a *repeat factor*.

¹¹ Alternative just diatonic scales might use 10/9 for the M2nd (which occurs between the 2nd and 3rd and 5th and 6th scale degrees), or 27/16 for the M6th. While these alterations shift the “wolf 5th” they have no effect on the central problem addressed here.

¹² Isacoff calls it “the search for *la*,” [p. 58] and constructs the following metaphor for the problem: “Attempting to force [pure thirds, perfect octaves, perfect fifths] them all into the same harpsichord or organ would be like trying to squeeze various pieces of fine furniture into a room too small to accommodate them; no matter where you place the exquisite couch, there is simply no room for the the elegant loveseat.... The keyboard thus became a battleground of warring proportions; and the desire to achieve a tuning based on all of the ideal, simple ratios an unrealizable fantasy.” [p. 67] He writes that musical temperament is, in “application... similar to the Japanese art of bonsai.” [p. 95].

¹³ A *comma* is, in terms of our framework, a more general idea, and roughly means “the error of an actual interval to an ideal interval.” Kennan suggests an algorithm for “optimally distributing” commas to “minimize the maximum of the absolute values of the errors of all the intervals that we care about.”

¹⁴ Personal communication, Polansky and James Tenney. Note that Barbour [1948 p. 550] seems to mean something very similar when he refers to “the tuning problem.” Doty [pp. 36–38] more specifically refers to the “supertonic problem”: “The problems posed by the supertonic minor triad and the syntonic comma have provoked some rather extreme reactions from composers and theorists.” [Doty p. 38]. The ideas in this paper are one such “reaction” to the generalization of these problems.

¹⁵ “Thus there will never be a scale in which all the fifths, or a complete set of fifths and thirds, are correct. The same type of analysis shows that any method of constructing a twelve-tone scale by rational numbers is doomed to inconsistency.” [Dunne and McConnell p. 109]. Several authors, including the latter, have employed the mathematical technique of *continued fractions* in this regard; see [Berger; Brun; Barbour 1948; Rosser]. Dunne and McConnell [p. 114] even develop a 41-tone to the octave tuning using this approach.

¹⁶ Jorgensen [p. 779] defines “wolf” quite generally, as “an interval that is considered too far out-of-tune to use in musical performance...its ratio is complicated.”

¹⁷ For the complete matrix, and detailed analysis of the “supertonic problem,” see Doty [pp. 36–38].

¹⁸ There are any number of definitions, in music theory, cognition, and history for terms like *scale*, *tuning*, *mode*, *tuning system*, *gamut*. Different authors mean different things and use these words more or less interchangeably. A discussion of the usages of these terms is beyond the scope of this paper, but in this example we refer to the Just Diatonic as a *scale* because it is a subset of a larger set of intervals. Some of these, like the ½-step 16/15, and the m3rd 6/5, do not explicitly appear in this representation. They would if the

scale were written as adjacent (rather than absolute) intervals: 1/1 9/8 10/9 16/15 9/8 10/9 9/8 16/15. The intervals do appear in the $\frac{1}{2}$ -matrix, but share diagonals with other intervals of different categories: both the $M2^{\text{nd}}$ (9/8) and the $m2^{\text{nd}}$ (16/15) appear in the same diagonals. The interval matrix of the larger tuning system from which the Just Diatonic scale is derived, would more clearly distinguish intervals.

¹⁹ Rasch says that “Few if any historical tunings have received more attention in the literature than ‘Werckmeister III’” [p. 29]. Donahue [p. 26] says of W3: “It may be the first documented temperament for keyboard instruments that did not have a wolf interval.” Similarly, our framework is intended to produce tunings without any “canidae intervals.”

²⁰ In these preliminary examples, it suffices to show the $\frac{1}{2}$ -matrix, although in our mathematical framework, for computational reasons, the error function is taken on the complete matrix. The interval matrix is symmetric around the repeat factor. Values below the diagonal are intervals that span the repeat factor until the next occurrence (in this case, two octaves of the tuning), so that each “key” is represented by the same number of intervals.

²¹ The L^1 and L^2 measures are but two in a continuous range of L^p measures, defined by taking the p^{th} root of the sum of the p^{th} powers (for any positive real number p) of the absolute value of the differences. Both the L^1 and L^2 functions measure pitch distance, but other error functions, such as those which measure the harmonic distance of the intervals (like the Euler *gradus suavitatis* (GS) function, the Tenney *harmonic distance* function [Tenney, 1988; see Chalmers [1983] for a comparison of these two functions], or the Barlow *harmonicity* function [Barlow] are possible as well. These functions suggest interesting ways to extend the framework in this paper [see, for instance Gräf].

A second possible extension to the framework would be to facilitate the choice of several weighted alternatives for certain ideal intervals, such as the aforementioned $M2^{\text{nd}}$ in the Just Diatonic scale. For example, the error function might use the minimum error to the 10/9 or 9/8, rather than the actual error to one of them.

²² In all cases tested, the genetic-algorithm solution with the L^1 measure finds the same solution as the deterministic one with the L^2 measure. However this non-deterministic approach offers the important possibility of easily substituting alternative error functions, and experimenting with other aspects of the framework. [See Software].

²³ This is due in large part to the particular construction of our framework, which incorporates redundancy. Each cell in the matrix is weighted by the product of the key and interval weights, allowing for (we hope) a more general and musically common specification of tuning system features. However, it is a simple matter to alter the way these weights are specified. For example, a *weight matrix* might be used, specifying a specific value for each cell. In this way, a greater degree of weighting specificity could be achieved, although, in many cases, at the expense of creating a less useful model.

²⁴ Tuning systems are often generated from a small number of important intervals, like the 5th and 3rd. Some preliminary experiments with our framework have shown that a number of historical and world music scales can be generated from small sets of ideal intervals, weighting all others to zero.

²⁵ For example keys beginning on the 1st, 4th, and 5th notes (*tonic*, *subdominant* and *dominant*) are thought to be “central” or inner in tonal music, an assumption usually

reflected in the tuning systems themselves. Similar hierarchies exist in other musical cultures, such as in Central Javanese music.

²⁶ Chalmers suggests a least-squares fit for the minimization of specific interval error, an idea influential to this current framework. Sethares [1997 pp. 69–70], introducing his own tuning optimization algorithm [also Sethares 1993] based on the best-fit to particular spectra, surveys a number of tuning evaluation methods (aptly titled “My Tuning is Better Than Yours”).

²⁷ Given some fixed number of pitches and of ideal intervals, it can be proved that ET achieves the minimum mean-tempering value, but so do other tuning systems. [See Appendix].

²⁸ Given a set of ideal intervals, our framework can generate infinitely many scales, including preexisting ones (like W3). For this reason it is worth clarifying the methodology of these examples.

We specified a small subset of ideal intervals (OWT1 = [5/4 4/3 3/2 8/5]; OWT2 = [9/8 7/6 5/4 4/3 3/2 8/5 12/7 16/9]) and searched for an accompanying *set of weights* (key and interval) which generated an optimal scale with the specified minimal *mean-tempering value*. To perform this search, we used a standard *hill-climbing algorithm* (in the program Matlab).

The weights found by the search algorithm were:

OWT1

interval = [1 1 1 50 1600 1 1 50 1 1 1]

key = [50 60 7 1 7 60 50 60 7 1 7 60]

OWT2

interval = [1 1 1 10 2000 1 2000 10 1 1 1]

key = [50 30 10 30 1 30 10 30 50 1 10 1]

²⁹ Several theorists and composers have discussed the importance and prevalence of septimal tunings. For example see [Erlich; Fokker; Harrison; Doty; Partch]. “So it seems natural to try and expand harmony by making 7-limit intervals as fundamental harmonic units. The new consonant intervals would be 7:4, 7:5 and 7:6.” [Erlich p. 2]. Doty [pp. 47–54] discusses some historical and “ethnic” seven-limit tunings. “Thus, while it may not seem accurate to say that no dominant seventh chords or diminished triads exist in the five-limit, the seven-limit versions of these chords offer such superior consonance and clarity that they are to be preferred in almost every case...” [Doty p. 44]. Fokker’s widely used 31-ET tuning has, in fact a more accurate approximation of what Rasch [Fokker p. 27] calls the “harmonic seventh” than the 3/2 P5th. Polansky [1984], citing Harrison [p. 114] and Surjodiningrat et. al., discusses septimal intervals in slendro gamelan tunings.