OWT: Real-time Software for Optimal Tuning Systems

ICMC 2008

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Five Constraints

Tuning systems through history and across cultures have used a set of complex compromises to account for some or all of the following constraints:

1. *Pitch set*: use of a fixed number of pitches (and consequently, a fixed number of intervals);
2. *Repeat factor*: use of a *modulus*, or *repeat factor* for scales, and for the tuning system itself (i.e., something like an octave);
3. *Intervals*: an idea or set of ideas of correct or ideal intervals, in terms of frequency relationships;
4. *Hierarchy*: a hierarchy of importance for the accuracy of intervals in the system;
5. *Key*: a higher-level hierarchy of the relative importance of the "in-tuneness" of specific scales or modes begun at various pitches in the system.

Comment: Most tuning systems attempt to resolve some or all of the five constraints listed above. These five constraints can be stated formally and mathematically, and constitute an economical and musically reasonable set capable of providing an interesting analysis of any tuning system.

Just Intonation and Well-temperament

Examination of a few well-known constructs from historical European tuning theory can illustrate some of the motivations for the mathematical framework.

Standard Just Diatonic scale (begun arbitrarily on "C"):

1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	(ratios)
C	D	E	F	G	A	B	C	(note names)
0¢	204¢	386¢	498¢	702¢	884¢	1088¢	1200¢	(cents values of intervals)

Central ideas in rationally based tuning systems

- *collision of primes*
- *"historical tuning problem"* ($p^n \neq q^m$ for distinct primes p and q (and $n, m > 0$))
- *"canidae" interval*

Interval Matrix

The *interval matrix* is a useful way to encode a tuning system.

	C (1/1)	D (9/8)	E (5/4)	F (4/3)	G (3/2)	A (5/3)	B (15/8)	C (2/1)
C		9/8	5/4	4/3	**3/2**	5/3	15/8	2/1
D			10/9	32/27	4/3	**40/27**	5/3	16/9
E				16/15	6/5	4/3	**3/2**	8/5
F					9/8	5/4	45/32	**3/2**
G						10/9	5/4	4/3
A							9/8	6/5
B								16/15

Interval Matrix of Just Diatonic Scale. Intervals are in ratios within one octave. 5^{th}s are in bold. The wolf (40/27) is the only non-ideal $P5^{th}$. Note that all $M3^{rd}$s (C-E, F-A, G-B) are ideal intervals of 5/4.

Ideal Tuning

An *ideal tuning* would be one in which the *i,j* entry only depends on |*i-j*| — each entry is equal to an *ideal* interval.

In the *ideal interval matrix*, values on the diagonals are constant and equal to the ideal ratio. The ideal interval matrix is equivalent to the interval matrix *only* in ET.

Error Matrix

The *error matrix* of a tuning system contains the differences between the entries of the interval matrix and the respective entries in the *ideal* interval matrix.

	C	**C#**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**C**
C		*90*	*192*	*294*	*390*	*498*	*588*	*696*	*792*	*888*	*996*	*1092*	*1200*
C#			102	204	300	408	498	606	702	798	906	1002	1110
D				102	198	306	396	504	600	696	804	900	1008
Eb					96	204	294	402	498	594	702	798	906
E						108	198	306	402	498	606	702	810
F							90	198	294	390	498	594	702
F#								108	204	300	408	504	612
G									96	192	300	396	504
G#										96	204	300	408
A											108	204	312
Bb												96	204
B													108

Figure 2: **W3 ½-matrix**. Each diagonal is a specific interval. "Keys" correspond to rows. All keys in W3 are considered reasonably good. Values are in cents.

	C	**C#**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**C**
C					4	0		6					
C#						22	0		0				
D							10	6		6			
Eb								16	0		0		
E									16	0		0	
F										4	0		0
F#											22	6	
G												10	6
G#													22
A													
Bb													
B													

Figure 3: **W3 error 1/2 - matrix**. Three (diagonal) intervals (M3rd, P4th, P5th) are shown for the ½ - matrix. Note that the "central keys" (C, F) have smaller errors for the 3rd and 5th (another central key, G, is the inversion of F, and is not shown in the ½-matrix).

Mathematical Formulation

The five constraints on tuning systems are formalized as follows:

1. *Pitch set*: let a_1 to a_n be a set of n pitches, none equal to 0.
2. *Repeat factor*: let $\omega > a_n$ be the repeat factor of the tuning system.
3. *Intervals*: let I_1 to I_n represent the ideal intervals.
4. *Hierarchy*: let i_1 to i_n be *interval weights* to represent the desired accuracy of the n intervals in the tuning system.
5. *Key*: let k_0 to k_n be *key weights* to represent the fixed pitches in the tuning system to which intervals are measured.

Interval, Ideal Interval, and Error Matrices

The *interval matrix M* for a set of *n* pitches, a_1 to a_n, is written as:

$$M = \begin{pmatrix} 0 & a_1 & \dots & a_{n-1} & a_n \\ \omega - a_1 & 0 & \dots & a_{n-1} - a_1 & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega - a_{n-1} & \omega + a_1 - a_{n-1} & \dots & 0 & a_n - a_{n-1} \\ \omega - a_n & \omega + a_1 - a_n & \dots & \omega + a_{n-1} - a_n & 0 \end{pmatrix}.$$

The *ideal interval matrix L* represents the desired interval for each entry in the matrix *M*:

$$L = \begin{pmatrix} 0 & I_1 & \dots & I_{n-1} & I_n \\ I_n & 0 & \dots & I_{n-2} & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_2 & I_3 & \dots & 0 & I_1 \\ I_1 & I_2 & \dots & I_n & 0 \end{pmatrix}.$$

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The *error matrix* is the difference between the interval matrix *M* and the ideal interval matrix *L*.

The total error of a tuning system is defined as a function of the error matrix. In the absence of any key or interval hierarchy (i.e., the interval weights and key weights are all equal) the error function is:

$$E(\vec{a}) = \sum \left(M - L\right)^2,$$

where the vector $\vec{a}$ contains the *n* pitches, a_1 to a_n, and the exponentiation and summation are applied element-wise to the matrix.

Interval and Key Weights

To formalize the notion of the relative importance of intervals and keys, we use interval and key weights. These weights can be applied to the error function through a weight matrix *W*.

The weighted version of the error function is

$$\hat{E}(\vec{a}) = \sum W * (M - L)^2$$

or

$$\hat{E}(\vec{a}) = \sum_{i=0}^{n} k_i \left[\sum_{j=0}^{i-1} i_{n+j-i+1}(\omega + a_j - a_i - I_{n+j-i+1})^2 + \sum_{j=i+1}^{n} i_{j-i}(a_j - a_i - I_{j-i})^2 \right]$$

Unique Solution

For each set of constraints specified in the matrices *I*, *K* and *L*, there is a unique solution that minimizes the weighted error function. This solution is called the *optimal tuning system*, and it is a set of pitches a_1 to a_n.

While the optimal tuning is unique to a given set of constraints, the converse is not true: there is not necessarily a unique set of constraints that will generate a given tuning. In other words, multiple sets of constraints can generate the same tuning, within a specified tolerance.

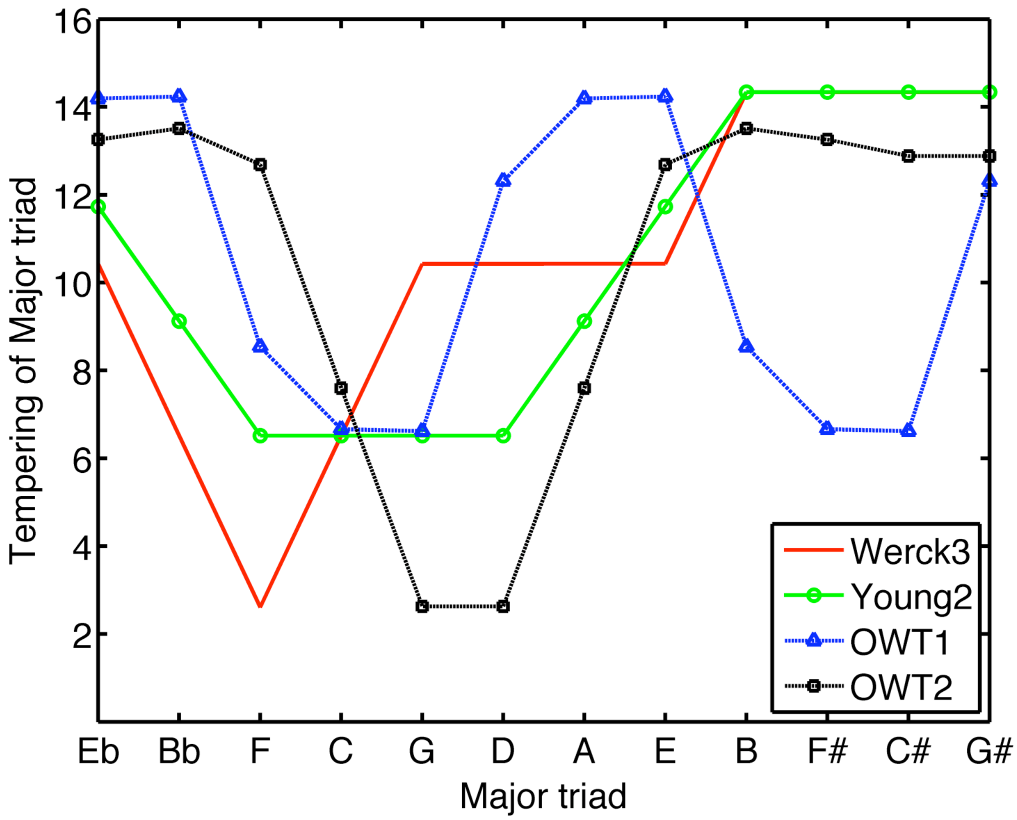
Minimal Mean-Tempering

We generated two new optimal tunings with the same *minimal mean-tempering* as W3, Young 2, and 12-ET. Using “reasonable” sets of ideal intervals, we found sets of weights for two new optimal tunings (OWT1, OWT2).

These two new tunings are *maximally in tune* by a specific measure: *mean-tempering of triads*. They have a great deal in common (ideal intervals, key and interval weights), theoretically and musically, with historical WTs. Yet, their musical implications and structure differ in important ways from their historical models.

W3	0	90.2	192.2	294.1	390.2	498.1	588.3	696.1	792.2	888.3	996.1	1092.2
Young 2	0	90.2	196.1	294.1	392.2	498.0	588.3	698.0	792.2	894.1	996.1	1090.2
OWT1	0	102.0	203.8	297.2	396.3	498.1	600.0	702.0	803.8	897.2	996.3	1098.1
OWT2	0	93.1	203.1	296.3	397.4	498.5	591.7	701.6	794.8	903.4	997.4	1091.4

Four different minimally-tempered WTs



Tempering of triads in four minimally-tempered WTs.

Future Directions

- *Further exploration of parameter space* (what is its geometry)
- *Constraint based system* —interest and veracity are dependent upon the choice of constraints/
- *Different error functions*
- *Multiple interval representations*
- *Exploration of tuning systems throughout the world*, using available documentation (such as Central Javanese slendro)

More examples, software, materials, soundfiles, writings:
http://eamusic.dartmouth.edu/~larry/owt/