Introduction

1.0 This paper is concerned with two related questions. First, it presents a methodology for the examination of intervallic relations as compositional determinants in atonal music. Secondly, in applying that methodology, it examines the value of computer utilization in such an analytic endeavor.

By "atonic" music we refer specifically to those works composed by Arnold Schoenberg, Anton Webern and Alban Berg between the years 1908 and 1923. Though fairly few in number, these works were radical in their implications and far-reaching in their consequences. In abandoning the practice of traditional tonal composition, Schoenberg renounced the most highly developed system in the history of western music. It was not
in Atonal Music

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until a decade and a half later that he formulated consistent and systematic procedures for the structuring of non-tonal composition. The structure of the works composed in the interim cannot therefore be understood either in terms of the tonal or twelve-tone systems.

Lacking any explicit statement defining structural criteria in these works, many have concluded with Perle that:

Aside from [the assumption of the semitoneal scale], the fundamental conditions of atonality are negative, merely stipulating the absence of a priori functional connections among the twelve notes of the semitoneal scale. (Perle, SC & A, 1).
Some critics have gone so far as to condemn these works as altogether lacking any positive constructive procedures (Cf. Reti, TMM). That certain compositional decisions may be based on negative criteria has been indicated by Schoenberg himself:

...an emphasized tone could be interpreted as a root, or even a tonic; the consequences of such an interpretation are to be avoided. Even a slight reminiscence of former tonal harmony would be disturbing, because it would create false expectations of consequences and continuations (Schoenberg, S & I, 108).

But to claim that such purely negative considerations comprise the "fundamental conditions" of coherent compositions would seem a serious overstatement.

This paper describes and analyzes certain organizational procedures within individual works. It is further hoped that the study initiated here may indicate further hypotheses concerning the existence of generally applicable structural criteria in the atonal works of Schoenberg, Webern and Berg.

It is unfortunate that most previous study of these works has yielded so little information of value. Some analysts have interpreted these works in terms of post-Wagnerian chromatic tonality (e.g., Leichtentritt, Neighbour). Others, such as Leibowitz, have indulged in somewhat vague philosophical generalizations concerning the historical development and aesthetic significance of "Schoenberg and his School". Leibowitz and, more recently, Perle, have dealt at some length with specific compositional techniques found in various atonal works. However, virtually all of their analyses have described only isolated examples of individual compositional techniques. Neither theorist has suggested or employed any unified systematic analytic procedure, and Perle has cast doubt on the efficacy of such procedures by stating that:

The "free" atonality that preceded dodecaphony precludes by definition the possibility of a statement of self-consistent, generally applicable compositional procedures. (Perle, SC & A, 9)

The most detailed and systematic published analysis of an individual atonal work, known to me, is Allen Forte's study of Schoenberg's Op. 19 (Forte, PNM). Most significantly, this article develops an analytic procedure on the basis of set-the-
oretic concepts, somewhat similar (but by no means identical) to their employment in twelve-tone theory. Forte’s article demonstrates the power of these techniques as applied to this particular composition, but is not concerned with other works of the atonal literature.

Therefore, to this date, no systematic analytic approach to this problematic body of compositions has been made. This article is an attempt to formulate a methodology for the study of a specific and limited aspect of atonal composition. It deals primarily with the determination of structural criteria with respect to the intervallic content of unordered “harmonic” collections of pitch classes.

The primary objective in the study of the harmonic theory of any musical system would seem to be the determination of syntactic constraints upon harmonic structure. However, prior to discovering the relations which govern the interaction of the elements of a system it is first clearly necessary to identify the elements, and basic collections of the elements themselves. Furthermore, beyond identifying the entire range of collections utilized in a given system, it is imperative that the normative collection (or collections) be identified and defined. Such establishment of the basic vocabulary does not necessarily insure an understanding of the syntax by which elements and their compounds are structured into a compositional whole. Here we recall that the tonal system, with its familiar norm, the triad, and its normative functional relation, the I-V pair, was given an adequate syntactic description only with the work of Schenker.

The present study does not attempt a similar task with respect to the harmonic structure of atonal music. It describes a method by which the normative “harmonic” collections and certain relations between these may be determined, a method which may eventually help to determine whether Schoenberg was correct when he wrote concerning atonal harmony that:

...I, and those who write in a similar way, distinguish exactly among those situations which call for a five- or six-voice (or more) chord. Without destroying the effect it would not be possible to omit a tone from an eight-voice chord or to add one to a five-voice chord. ... If one tone is changed the chord is different; perhaps it would be appropriate in another location, in another relation. It appears that laws operate here — I know not which laws. Perhaps I will know in a year or so; perhaps
someone else will discover them. At the moment we can only describe. (Schoenberg, Harmonielehre, 470)

Most previous studies of this music have avoided detailed analysis of interval content. This is perhaps largely due to the somewhat tedious counting and computing operations which such a systematic study entails. The application of the computer in this task therefore seemed highly appropriate. In this way, a large sample of works could be analyzed in considerable detail. Thus, with the aid of the IBM 709 Data Processing System at the Yale Computer Center, a total of 19 distinct pieces, (from four separate works) were analyzed with respect to intervallic content of phrase groupings. Over 1000 such "compositional sets" were input and their interval content (as defined below) computed. Within each piece, each set was compared with each other set, on the basis of certain pre-defined relations. Statistical tallies and arithmetic means (described in the last section of this paper) were also computed.

1.1 Computer applications

The investigation of the advantages and disadvantages of the utilization of a computer in musical analysis was a significant concern throughout the project. Several previous applications of the computer in the domain of music have been attempted, the most extensive work having been done by L.A. Hiller and his associates at the University of Illinois (Hiller, ExM). Although Hiller has suggested certain analytic applications (entirely unlike the present study) his published work to date has been concerned primarily with the computer synthesis of musical structures. An interesting, but musically naive analysis-synthesis project was carried out at the Harvard Computation Laboratory several years ago (Brooks). This study analyzed the melodic succession of a sample of 39 hymn tunes on the basis of Markoff analysis of up to eighth order, and then synthesized similar melodies utilizing probabilistic constraints derived from the Markoff analysis of the initial sample. The results are musically crude, though the study has other interesting (non-musical) implications. One or two other studies have utilized digital computers in musical studies (primarily from the perspective of information theory), but no study has dealt directly with the harmonic or intervallic materials of "problematic" compositions. It therefore seemed of interest to experiment with such a computer technique and to draw some conclusions as to its present (as well as future) applicability in this field.
Perhaps the most serious limitations on the computer's flexibility in the musical domain lie in the realm of input since no existing computer language can adequately describe the complex aspects of a musical score.*1 In addition, the computer's limitations in identifying and selecting certain patterns from a large mass of input data necessitated "pre-analysis" of compositions by hand in order to define the sets of PCs to be analyzed. A great amount of work is now being done in the area of computer pattern recognition, and also in the development of "scanners" which would be able to read directly from score (Hiller, EXM, 176). The use of such devices would eliminate the most tedious aspects of computer usage: coding (i.e. translating musical notation into numerical equivalents) and card punching. It is doubtful, however, that in the near future a sophisticated enough program could be written to compensate fully for the trained musician's intuitive abilities in determining significant musical units.

Fewer problems were encountered in the computation program which was written to duplicate effectively certain analytical devices which were thought to be significant for the present study. Writing this program was time-consuming, but, having now been completed, it may be utilized in the analysis of any number of compositions, including that of certain twelve-tone techniques. Because of this general relevance, the outlines of the program will be followed in the explanation of the analytic procedures below. But first, the basic terminology and procedures of the analytic method must be defined and explained.

2.0 Basic definitions

The traditional set of available pitch material whose elements are the twelve pitch classes (henceforth abbreviated PC's) of the semi-tonal scale, "with class membership defined by octave equivalence" (Babbitt, MQ, 247) is assumed. For notational convenience, these twelve (equal tempered) PC's are placed in one-to-one correspondence with the set of non-negative integers \( \{0, 1, 2, \ldots, 11\} \) such that:

\[
\begin{array}{cccccccccccc}
C & C\# & D & D\# & E & F & F\# & G & G\# & A & A\# & B \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}
\]

Any pair of non-identical PC's may comprise an interval. We follow the traditional concept regarding the magnitude of an interval, defining it as \( |PC_a - PC_b| \). It may be noted that by defining the magnitude of an interval in terms of the absolute
difference between $PC_a$ and $PC_b$, we have in effect denied the significance of PC order in the determination of intervallic magnitude. This magnitude is clearly invariant under any equal transposition of both members of the pair of PC's. We further adopt the familiar notion of interval inversion, such that (for instance) a minor 2nd is considered equivalent to a major 7th. We therefore define six interval classes, $IC_1, \ldots, IC_6$. Intervals may be said to belong to the same $IC_j$ ($j = 1, \ldots, 6$), if their magnitudes are equal to $j$ or are equivalent to $j$ by the defined relation of complementation mod. 12. *2

The above basic notions enable us to consider all the unordered pairs of elements formed by a particular set of PC's as a "harmonic" collection. By considering each of these pairs in a given set of, for instance, 4 PC's, we compute the number of occurrences of each of the six possible ICs which may be present in that set. Thus, for the PC set $(0, 4, 7, 11)$ the IC occurrences may be listed systematically as follows:

<table>
<thead>
<tr>
<th>PC's</th>
<th>Interval</th>
<th>IC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 4)</td>
<td>0-4</td>
<td>4</td>
</tr>
<tr>
<td>(0, 7)</td>
<td>0-7</td>
<td>5</td>
</tr>
<tr>
<td>(0, 11)</td>
<td>0-11</td>
<td>1</td>
</tr>
<tr>
<td>(4, 7)</td>
<td>4-7</td>
<td>3</td>
</tr>
<tr>
<td>(4, 11)</td>
<td>4-11</td>
<td>5</td>
</tr>
<tr>
<td>(7, 11)</td>
<td>7-11</td>
<td>4</td>
</tr>
</tbody>
</table>

$(0, 4, 7, 11)$

We term the distribution of IC occurrences associated with a given set of PC's the interval content of that set. This interval content may be conveniently represented by an ordered six-tuple, the $i$th component of which represents the number of occurrences of $IC_i$. In this positional notation, the interval content of the previous example would be represented: $[0 1 2 0 0 1]$. We will henceforth refer to such a six-tuple as the "interval vector of the set $(0, 4, 7, 11)$".*2 Throughout this paper, PC sets will always be enclosed in parentheses ( ), whereas IC vectors will be enclosed in square brackets [ ].

In defining a theoretical model for analytical purposes, certain reductive criteria must necessarily be applied. In addition to the familiar principles of octave equivalence and intervallic complementation noted above, the following criterion will be operative throughout the present study. In computing the interval vector of a compositional statement of a PC set, we
delete any repetition of a PC (or PC's) from the set; that is, PC's which occur more than once are considered as a single occurrence of the PC. This restriction is in accord with the customary definition of a set which states that

... the same element shall not be allowed to appear more than once. The number complex 1, 2, 1, 2, 3, consequently becomes a set only after deleting the repeated elements. (Kamke, 1)

This restriction was necessary in our analytic method in order to limit the number of possible intervallically distinct PC sets.

We further make the following delimitations: We eliminate all sets containing 1 PC since (obviously) no IC set may be associated with such sets. Sets containing 11 and 12 PC's are trivial cases and are also eliminated, since no more than one intervallically distinct set may be associated with sets of these magnitudes. That is, for all 12 possible sets of 11 distinct PC's, the interval content would be represented by the vector: [10 10 10 10 10 5]. For the one set of 12 PC's, the interval content would be represented by the vector: [12 12 12 12 12 6]

On the basis of the above restrictions we may distinguish 197 unique sets of IC's, containing from 1 to 45 IC occurrences, and each associated with a unique PC set (disregarding transposition) which may contain from 2 to 10 elements. Given a set of n PC's, the number of intervals in the associated set is uniquely determined by the binomial coefficient \( \binom{n}{2} \). In our vector notation, the total number of intervals contained in a given set may easily be determined by summing the number of occurrences of each interval class in the vector:

\[
\text{PC set } = (0, 1, 2, 3) = \text{IC Vector } = [3 2 1 0 0 0] \text{ total number of interval occurrences } = 6
\]

A complete list of all the possible unique IC sets (containing from 1 to 45 intervals), each with its associated PC set is given in Forte, JMT, 145f.*3 Each PC set is ordered in so-called normal form:

... wherein the interval determined by the first-last p.c. is greater than any interval determined by successive p.cs in [the set]. (Babbitt, JMT, 77)

In referring to any set contained in the tables, we will use the notation (n-m), where n refers to the cardinality of the PC set,
and $m$ refers to the order position of the set on the list. The designation $(4-1)$ therefore refers to the first set on the list of sets containing four PC's.

Properties of certain of the sets with respect to the twelve-tone system are also discussed by Donald Martino (See Martino, 224ff.) See also Lewin (JMT III & IV), Perle (Score, and SC & A, 197ff.).

Having defined the elements, and described certain of the relations and operations of our analytic procedure, we now proceed to describe the application of this analytic method to a sample of atonal musical compositions.

Application of the Analytic Method

3.0 Summary description:

A sample from the atonal works of Webern and Schoenberg was chosen. On the basis of traditional musical criteria (described below) these works were subdivided into "vertical" or "horizontal" sets containing from 2 to 10 pitch classes. These compositional PC sets were then numbered sequentially for identification purposes, coded in numerical notation (described in section 2.0), and input to an IBM 709 digital computer. The program instructed the 709 to compute the interval content of each compositional set, to store this information in the "interval vector" form (described above) and then to compare each pair of vectors in a given composition. As each compositional set was compared with each of the other sets in the work, each pair of intervallically identical sets was recorded. Intervallic identity having been established, the relation between the two PC sets was examined to determine whether these sets were related by the operations of (unordered) transposition or (unordered) inversion. Set pairs which could be regarded as both "T" and "I" related were so identified. In those few cases in which two PC sets may be intervallically identical but are not "T" or "I" related, they were designated as "Z-related". (The properties and several compositional occurrences of these special sets are discussed in sections 3.7 and 4.2, below).

All pairs of compositional sets of the same magnitude which were not found to be intervallically identical were then compared to determine the degree of similarity of intervallic distribution. The measure by which this similarity relation was determined is described below, in the explanation of SUBROUTINE DIFR. A separate subroutine was also written which
identifies ordered relations between PC sets both of equal and unequal magnitudes, but this subroutine was considered to be somewhat beyond the scope of the present investigation and is only briefly described in this paper.

The printed output lists and numbers all the compositional sets in the order of their occurrence in each piece. The interval content of each set is printed immediately below the PC numbers of the set, and the identification numbers of all compositional sets which are related to a given set (both in terms of intervallic identity and distributional similarity) are printed with that set. Designations of T, I, or Z are also made for sets so related. Related sets are cross-referenced so that if, for instance, set 20 appears under set 1 as some relation, set 1 is also printed under set 20.

Because of the difficulty in interpreting the entire output (which exceeded 25,000 printed lines), a statistical routine was written to summarize the data. The results of this statistical analysis are represented in the graphs of Figures I-XVI, which are explained in section 5.

3.1 Choice of compositions

Webern's atonal works were considered to be all those from Op. 4 through Op. 16. From these we selected a random sample of three opera which therefore represented a significant percent of the total population (229 measures out of approximately 1200). The works chosen were the Five Movements for String Quartet, Op. 5, the Three Pieces for Violoncello, Op. 11, and the Four pieces for Violin, Op. 7. The sample therefore yielded a total of 12 individual pieces.

3.2 Determination of compositional sets

As mentioned previously (section 1.1), some pre-analysis was necessary for the determination of the compositional sets. The criteria which guided these selections were primarily traditional musical ones. Two broad categories of compositional sets were defined: horizontal (melodic) sets and vertical (simultaneous) sets. The criteria utilized in defining the horizontal compositional sets included:

1 Phrase indications (slurs) and other articulation markings indicating modes of attack.
2 Durational considerations, particularly the use of rests to demarcate the ends (and beginnings) of "melodic
lines”.
3 Timbral disjunction, both instrumental and coloristic (sul pont., pizz., etc.).
4 Dynamics, particularly the characteristic “hairpins” and diminuendo phrase endings.

Vertical sets were determined largely by those criteria which usually are considered in defining any simultaneity as a “harmonic unit” (or more specifically a “chord”) in traditional terms. Factors such as the following were considered:

1 Simultaneous attack of two or more PCs, (though this was obviously not a strict requirement for the identification of a vertical set).
2 Significant “effective” duration, approximating at least 1 metrical unit.
3 Temporal disjunction from preceding and succeeding material by means of rests.
4 Timbre
5 Dynamics

Transient simultaneities arising from independent contrapuntal lines were not generally considered to comprise vertical sets. Naturally, a host of intuitive factors perform an important function in such decisions as these. In selecting the compositional sets it was felt that maintenance of a general consistency of criteria, tempered with some flexibility, was preferable to rigidly objective, but perhaps unmusical restrictions. Thus, for instance, although timbral disjunction was a general criterion for PC set disjunction, certain passages involving “Klangfarbenmelodie” necessitated exceptions to this rule.

Obviously the determination of a musical phrase is in some measure a subjective decision, and the interpretation of problematic phrasing may vary among musicians. To compensate for this problem in coding, all passages in which the phrase groupings seemed ambiguous were coded in several possible interpretations. For instance, a melodic line which might be further subdivided into two shorter phrases was coded as one long set and also as two short sets.

The validity of the choices of these compositional sets was informally “verified”: several experienced musicians were told the basic criteria listed above, and then asked to select compositional sets in several of these works. The results were strikingly similar to our initial selections.
3.3 The Program: MAIN

As indicated by the flow diagram (Example 1), the program consists of three sections: MAIN, SAME, and DIFR. In describing the various functions performed by the program we shall follow the flow of control as illustrated by the diagram. Programming details will necessarily be omitted from the description.

MAIN performs 3 primary functions. It controls the input of the data, (and also some of the output), computes the interval vectors for all input PC sets, and compares each pair of vectors to determine whether they are identical—in which case control passes to SUBROUTINE SAME—or non-identical—which causes control to pass to SUBROUTINE DIFR.

3.4 SUBROUTINE SAME

The function of this subroutine was to determine the existence of the transposition, inversion, or so-called Z operators between each pair of intervallically identical (unordered) PC sets. Whenever a comparison between IVEC (I) and IVEC (L) indicated that they were identical, the corresponding compositional PC sets KOMPS (I) and KOMPS (L) were read in to SUBROUTINE SAME. Since these sets were unordered, it was first necessary to arrange them in some consistent ordering for the sake of comparison.

3.5 Normal form algorithm

For this purpose a routine was written to arrange the PCs into normal form (Section 2.0). Although the programming details are too elaborate to be included here, I thought it valuable to include an informal description of the algorithm on which the program is based.

Given an unordered set of PCs, such as (0, 6, 10, 2), first rearrange the PC numbers in ascending order:

\[(0, 2, 6, 10)\]

Take the difference between the values of the last and first elements and store this value in a list. Then add 12 to the value of the first element and permute the set circularly so that the first element now becomes the last, the second becomes the first, the third becomes the second, etc. Then store the difference between the new values of the last and first elements, again add 12 to the first element and perform the
same circular permutation. Repeat this procedure as many times as there are elements in the set. Thus:

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>(l-i)</th>
<th>(j-i)</th>
<th>(k-i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>10</td>
<td>12</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>12</td>
<td>14</td>
<td>18</td>
<td>8</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

That unique ordering which contains the smallest value of (l-i) is termed the normal form of the set. If two or more set orderings display the same value of (l-i) as in the above case, we select the set which exhibits the smallest value of (j-i). If this value is still not unique to a single set ordering, we compare the values of (k-i), etc., until such a unique value is determined. In the above example the comparison of the jth and ith elements of the last two set orderings yielded a unique minimal difference of 2 in the last ordering. By the relation of congruence mod. 12 we subtract 12 from all elements whose magnitudes exceed 11, and thus define the normal form of PC set (0, 2, 6, 10) as the set ordering (10, 0, 2, 6). It should be mentioned that for the sake of uniformity, normal forms are transposed to begin on 0—in which case the latter example would equal (0, 2, 4, 8). This transposition was not carried out in SUBROUTINE SAME, since the primary purpose of the normal form algorithm was to arrange two unordered PC sets in an arbitrary (but corresponding) order so as to determine the transposition number which related them. This was the next step in the program. We continue to follow the flow as represented in the diagram of SAME.

3.6 Determination of T and I relations

Having computed the normal forms of both KOMPS (I) and KOMPS (L) we now may determine if these sets are transpositionally related simply by taking the absolute difference (mod. 12) between the PC numbers in the corresponding order positions of each set. If all such differences equal the same value, then that value equals the t (transposition operator) which relates the two sets. For instance
example
KOMPS (I) = ( 0,  6, 10, 2)
KOMPS (L) = (11,  7,  3, 1)
Normal Form KOMPS (I) = (10,  0,  2, 6)
Normal Form KOMPS (L) = (11,  1,  3, 7)
Differences = 1,  1,  1,  1, (mod. 12)

Therefore: KOMPS (L) = T₁ KOMPS (I)

As indicated on the flow diagram, once such a T-relation was established by the computer, the relation, the T number, and the number of KOMPS (L) were all printed out. Control then passed to a set of instructions which determined whether KOMPS (I) and KOMPS (L) were inversionally related. This was done by comparing the normal form of KOMPS (I) with the "inverted" normal form of KOMPS (L). Inverted normal form is derived by arranging the set in descending order, taking the difference between first and last elements and then adding 12 to the last element and permuting the set circularly so that the last element now becomes the first.

\[
\begin{array}{cccc}
i) & j) & k) & l) & m) \\
KOMPS (L) = & (1, & 3, & 2, & 11, 5) \\
\text{Descending order} = & (11, & 5, & 3, & 2, 1) \\
& (13, & 11, & 5, & 3, 2) & 10 \\
& (14, & 13, & 11, & 5, 3) & 11 \\
& (15, & 14, & 13, & 11, 5) & 10 \\
\text{Inverted N. F.} = & (17, & 15, & 14, & 13, 11) & 6
\end{array}
\]

We reduce this set to the congruent set (mod. 12) within the range 0, . . . , 11, and then compare it with a set in "ascending" normal form, by adding the PC numbers in the corresponding order positions:

KOMPS (I) = 7, 9, 10, 11, 1
KOMPS (L) = 5, 3, 2, 1, 11

Sums = 12, 12, 12, 12, 12 (=0, mod. 12)

If all the sums are the same (mod. 12) the two sets are inversionally relatable. Since this particular subroutine does not examine ordered relations between PC sets it does not deter-
mine whether two sets KOMPS (I) and KOMPS (L) are actually I-related in their compositional ordering. There are, of course, many cases in which two unordered PC sets may be related by both transposition and inversion. In such cases the program prints out both relational possibilities, and an inspection of the compositional statements of these two sets is necessary to determine which relation might be operative. Many instances of two intervallically identical sets which could only be related by (unordered) inversion were also revealed in the output.

3.7 The Z-relation

Finally, if neither I nor T relations could be established between two intervallically identical PC sets, they were designated as being in the "Z"-relation. These special cases have been noted by David Lewin:

If P and Q are collections of six or less notes, with identical intervallic contents. . ., then either

1. Q is a transposition or inversion of P or
2. P and Q are six-note collections and Q is a (transposed or inverted) form of the complement of P or
3. P (or Q) is a (transposed or inverted) form of the tetrachord C, D, F, F#, and Q (or P) is a form of the tetrachord B, C, D, F# or
4. P and Q are forms of the pentachords B, D, F, F#, G, and F, G, A♭, B, C or
5. P and Q are forms of the pentachord B, C, D, E, F, and C, C♯, D, E, G and these are the only possibilities. (Lewin, JMT, 1959)

Set pairs which are described in the second through fifth categories above have been defined as Z-related. It should be added that for collections of more than 6 PCs, there exists a corresponding complementary pair of Z-related sets for each of the tetrachord and pentachord pairs described above. The importance of these Z-related sets in the atonal works analyzed by the program will be discussed in the section dealing with the interpretation of the output.

Following the printing of the T-, I-, or Z-relations discovered between any KOMPS (I) and KOMPS (L), control returns to MAIN, where the value of L (and I) is incremented, and the next pair of IVECS are compared. If IVEC (I) and IVEC (L) are found to be non-identical control passes to SUBROUTINE DIFR.
3.9 SUBROUTINE DIFR

The purpose of this subroutine was to measure the degree of similarity (or dissimilarity) of intervallic distribution between any two non-identical interval vectors which described PC sets of equal magnitudes. This measure of intervallic similarity is termed the “similarity index” and is computed by the following formula:

\[
\sqrt{\sum_{j=1}^{6} (IVEC (I)_j - IVEC (L)_j)^2} = \text{Similarity Index}
\]

That is, given two ordered six-tuples, IVEC (I) and IVEC (L), take the difference between the corresponding entries, square each difference, sum the squares and take the square root of the sum:

IVEC (I) = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (4-1)

IVEC (L) = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \end{bmatrix} \quad (4-2)

differences = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}

\[
\sqrt{1^2+0^2+0^2+1^2+0^2+0^2} = \sqrt{2} = \text{s.i.}
\]

It will be readily seen that the greater the degree of distributional similarity the smaller will be the value of s.i. The s.i. for two identical IVEC’s will equal 0.

Musically, this measure appears to be a significant indicator of the similarity or dissimilarity of intervallic content between two harmonic collections containing an equal number of PCs. Thus whereas the s.i. for the two closely related sets shown above equalled \( \sqrt{2} \), the s.i. for two such intervallically dissimilar collections as the “diminished 7th chord” and 4-PC whole-tone scale segment equals \( \sqrt{30} \):

PC set = (0, 3, 6, 9) vector = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 2 \end{bmatrix} \quad (4-28)

PC set = (0, 2, 4, 6) vector = \begin{bmatrix} 0 & 3 & 0 & 2 & 0 & 1 \end{bmatrix} \quad (4-21)

\[
\sqrt{0^2+3^2+4^2+2^2+0^2+1^2} = \sqrt{30}
\]

\( \sqrt{2} \) and \( \sqrt{30} \) define the limits of the range of s.i. values for sets of 4-PCs. In general, for two sets of any magnitude (within the range of from 3 to 10 PCs) the maximum possible similarity equals \( \sqrt{2} \), whereas the maximum possible dissimilarity varies
from a low of $\sqrt{14}$ for sets of 3 and 9 PCs (complementary sizes are always in the same relationship) to a high of $\sqrt{72}$ for the sets of 6 PCs:

$$\begin{align*}
\text{PC set } &= (0, 1, 4, 5, 8, 9) \text{ vector } = 3 \ 0 \ 3 \ 6 \ 3 \ 0 \ (6-20) \\
\text{PC set } &= (0, 2, 4, 6, 8, 10) \text{ vector } = 0 \ 6 \ 0 \ 6 \ 0 \ 3 \ (6-35) \\
\sqrt{3^2+6^2+3^2+0^2+3^2+6^2 &= \sqrt{72}}
\end{align*}$$

Interpretation of Output, I: Contextual Analysis

4.0 As indicated in the general description of the project, the output was analyzed and interpreted in several different ways. Since the output format was designed to facilitate the analysis of individual works, a contextual analysis was undertaken first. Obviously the scope of the computer program was such that only a defined and limited aspect of the compositions was actually "analyzed" by it. Nevertheless, it was found that by establishing structural analogues on so seemingly significant a basis as intervallic content, the output data served as a valuable guide in interpreting other aspects of compositional practice. Certain compositional criteria were examined, such as those governing specific set transpositions, and the use of PC invariance or transformation as a means of "harmonic progression". Those aspects necessarily excluded from the computer program, such as durational value, timbre, and register were now correlated with the computer's analysis of intervallic relations.

Before presenting this contextual analysis, it was thought of interest to contrast certain of the uninterpreted data output by our computer program, with the description of aspects of Op. 5/1 made by a human analyst—Rudolph Reti. Reti's analytic remarks concerning this piece are offered in support of his thesis that atonal music lacks any "binding or form building principles" (TMM, 56) and that compositional decisions are largely "arbitrary", or even "random" (ibid. 94).

4.1 Webern: Op. 5/1

Among his other accusations concerning this piece, Reti states that, were it not for the PC invariance of C and C♯ between the opening and closing measures of the first section (mm. 1 & 6 — See Example 2), "one would even think of implying a relationship between the ending of the group and its beginning." (ibid, 97) Regarding the "harmonic accompaniment" in mm. 2-3, Reti states that the chords are in a "...random vein,
(example 2)
adding nothing to the compelling quality of the design" (ibid., 94). Finally, he complains that though a kind of "tonicality through pitches" is almost achieved at the end of the piece (by the repetition of C and C♯, which Reti takes to be the "tonical pitches" of this piece), "a pizzicato chord in pianissimo of different pitches is annexed, through which the hard-won quasitonal unity is annulled" (TMM, 97).

An examination of Example 2 will reveal certain discrepancies between Reti’s interpretation and the computer’s findings. Regarding Reti’s first statement quoted above, the computer analysis has more than remotely implied a relationship between the opening and closing of the first group: It has identified these two 4-PC sets as being intervallically identical. Furthermore, this symmetry between beginning and ending is repeated in the next section, which opens and closes with the same intervallic set as that which opened and closed the first section. The important recurrences of this set towards the close of the piece will be discussed below.

Concerning the "chords in a random vein" in mm. 2-3, clearly these "chords" are in fact only one chord, the last two simultaneities in m. 3 being simply transpositions of the initial statement in m. 2. It should be further noticed that this set reappears in a transposed linear statement in the second group (mm. 8-9) in a location analogous to its position in the first. This same melodic figure is then utilized to open the third section with an elaborate "stretto" (mm. 14-15). The intervallical ordering of the statement at mm. 8-9 is maintained in the stretto, and PC invariance also is evident.

With regard to the "annexed chord of different pitches" in the last measure of the piece, the computer analysis has shown this chord to be intervallically identical to the important chord which is first stated in mm. 5-6, and recurs frequently (and at significant structural points) throughout the piece. In terms of PC identity, the "different pitches" of this chord are identical to the statement of the same set which is presented both vertically and horizontally at mm. 17-18. Thus, even without further interpretation, the computer output reveals many significant set associations overlooked (or denied?) by Reti. Closer study of the score, with the guidance of the computer analysis reveals many more.

The technique of PC invariance appears to be a significant transpositional determinant in this work. Set (4-7) which opens the piece is transposed by 8 (T8) when it is restated at the
close of the first section (m. 6). Vertical spacing is not main-
tained however, and the fact that this particular transposition
holds C and C♯ invariant would seem to be the primary deter-
minant for choosing this t, since these two PCs are (as Reti
rightly points out) the most strongly and frequently iterated PCs
in the opening section of the piece. Regarding such PC iter-
ations it is interesting to note the following statement by Webern:

In this musical material new laws have come into force
which made it impossible to describe a piece as in one
key or another. . . we sensed that the frequent repetition
of a note, either directly or in the course of the piece in
some way "got its own back" — that is, the note "came
through". (Webern, 51)

The next statement of set (4-7) introduces a new section (m. 7)
and represents a T₃ of the original statement in m. 1. Neither
C nor C♯ is retained under this transposition and the elements
are grouped to emphasize two successive major 3rds rather
than the augmented octaves of the opening. This emphasizes
the formal disjunction between the first and second sections,
also brought out by the introduction of Tempo II and the low
register. The set statement in m. 6 therefore relates (by in-
variance of C and C♯) to the opening of the first section, and
the compositional set in m. 7 relates (by exact repetition) to
the closing of the second section at m. 13. Emphasis on C♯ is
still maintained by its appearance in a "new" intervalllic set
((4-3), cello, m. 7-8), and it is then associated once more with
C♯ in the melody stated by the first violin in m. 9.

More complex is the significance of the 4-PC set (4-12) in the
cello at mm. 8-9, and at mm. 14-15. As indicated in Exam-
ple 2, this set is intervallically identical to the vertical sets in
mm. 2-3. It should also be noticed that the linear statement
at mm. 8-9 (T₆) concludes with the PC's A♭-D which appeared
as a double stop in the viola in m. 2. As mentioned above, at
the close of the second section (m. 13), Webern repeats the
two sets with which he opened it. He then begins the next sec-
tion with a stretto of (4-12) which departs from the same PC's
(and same ordering) as the cello statement at mm. 8-9 (i.e.,
F, G♭, A♭, D in Vln. I). This passage actually begins as a
double canon, between Vln. I and Vla. and Vln. II and Cello,
each at the 8ve, one sixteenth apart. Most interestingly, not
only is the pitch content of the first group of four PCs in Vln.
I (and Vla.) identical to that at m. 8-9, but also the pitch con-
tent of the second group of four PCs is identical to that of the
initial statement of this set at mm. 2-3. The interval succes-
sion and contour of the linear statement is also maintained. This canonic texture rapidly disintegrates through the dispersion of the pitch motive between the different instruments. The transposition of the final complete statement of (4-12) returns to the pitch content of the opening of the passage (F in Vln. I, G\textsuperscript{b} A\textsuperscript{b} D in Cello), and of course to that of the statement at mm. 9-10. The first violin then begins a statement of set (4-3) which is T\textsubscript{3} of the statements at m. 7 and m. 13 and which clearly emphasizes the PC A\textsuperscript{b}. Set (4-7) is still associated with (4-3), though the positions (and instrumentation) are inverted. These sets are interrupted by a fff statement of the vertical set which first occurred at mm. 5-6. The passage which follows is one of the most complex and interesting sections in this movement, and may supply a clue to the interaction of interval and PC invariance as a compositional technique in Webern’s atonal works.

The 8-PC set (8-10) first stated as a simultaneous at mm. 5-6 has been identified by Perle as “... a symmetrical chord – that is, a chord that may be analyzed into two segments, one of them the literal inversion of the other” (SC & A, 27). The vertical restatement of this set at m. 17 is T\textsubscript{2} related to the initial verticality, and the spacing is identical. Most significantly, this particular T maintains invariance of all four PCs of the lower segment (marked in the score as x) of the initial statement (C, G\#, E\textsuperscript{b}, B), which now comprise the upper segment of the T\textsubscript{2} version. Thus the two statements of (8-10) are both comprised of two intervallically identical 4-PC sets (i.e., 4-17: [1 0 2 2 1 0]) and each contains the invariant 4-PC set (C, G\#, E\textsuperscript{b}, B).

In the following measure, the T\textsubscript{2} version of (8-10) is horizontalized and divided into its two component tetrachords. Segment y is stated by the cello, and x by the viola, in a stretto-like imitation a minor seventh higher. When this imitation is continued by the second violin at the seventh above the viola, the four PCs which are generated are exactly those of the upper segment w of the verticality in m. 5. Thus the common 4-PC segment x acts as a kind of pivot between these two transpositions of this 8-PC set. The entrance of Vln. I in m. 18 picks up the pizzicato motive G, B, A\textsuperscript{b} from the Vln. II, but by altering the final pitch from B\textsuperscript{b} (which is still prominent in Vln. II) to E, another transposed statement of (4-17) is effected. Combined with segment w in Vln. II, these segments comprise a further transposed statement of (8-10). However, the pitch content of the initial statement (w x) is further emphasized: the cello reiterates segment w in an ostinato pattern, while the
viola begins what appears to be a restatement of the motive set (4-3) – Cf. mm. 7-8, 13, 16-17 etc. – now at T_7 of the original statement. Although the first four PC’s of this linear set preserve both the ordered interval succession and melodic contour of the previous statements, the viola line continues to unfold additional PCs until it has stated seven of the eight PC’s comprising the chord at mm. 5-6. Thus the entire passage from mm. 18-22 might be thought of as a horizontal “composing out “of the vertical statements of (8-10). Through the use of PC identity the passage leads away from the content of the T_2 statement (segments x & y) at m. 17, through the pivotal use of the invariant content of segment x, back to a complete statement of the pitch content of the initial (w and x) segment as at mm. 5-6. The analysis of such a procedure as this suggests certain hypotheses concerning the interrelationship of interval and PC invariance in these works. (See section 5.2). With regard to the viola solo in mm. 19-22 it is interesting to note that if considered as a T_2 of the cello part at mm. 7-9, the only “added pitch” is the Bᵇ. Observe also that the 3-PC motive which ends the solo is identical to the “partial statement” of the stretto motive which ends the section at m. 16 (i.e., C, Dᵇ, Eᵇ).

The stretto in m. 47 contains several compositional techniques of particular interest. The stretto is based on the opening 4-PC set of the piece (4-7) which is now reintroduced as two verticalities in the lower strings (m. 46). The techniques of pitch transformation and invariance described above, may also be observed here. The first of these two verticalities contains none of the six PC’s contained in the following stretto (m. 47). This chord is then equally transposed up a whole step, which generates four new PC’s (Bᵇ, D, A, C♯). These PC’s are immediately taken up by the cello and viola, followed by Vln. I and II in stretto-like imitation. Further unity is obtained by the repetition of invariant ordered adjacencies between the transpositions of successive linear statements of (4-7):

Vln I: D, C♯, F♯, F♮ T₄: F♯, F♮, B♯, A
Vla. & Vln II: Bᵇ, A, D, C♯ T₄: D, D♯, Fᵇ, F♮
Cello: F♯, F, Bᵇ, A T₄: Bᵇ, A, D, C♯

It will be noticed from the above that in all of the transpositions, six PC’s remain invariant throughout. The reader with a knowledge of twelve-tone combinatorial procedures may be surprised to observe that the PC invariance here is maintained through “use” of the type E third order all-combinatorial hexachord (See Martino, 229). If we designate the initial statement of
this set in the viola as $T_0$, then the two imitative entries of the
first violin and cello represent complementary transpositions
of the hexachord which not only maintain invariance of the total
pitch content, but which maintain invariant ordered adjacencies
as well.

\[
\begin{align*}
T_4 &= D \quad (\text{vln. I}) \\
T_0 &= B \quad (\text{vla, vln. II}) \\
T_8 &= F^\# \quad (\text{cello})
\end{align*}
\]

\[
\begin{align*}
D &\quad C^\# & F^\# & F^\natural & B^b & A \\
B^b & A & D & C^\# & F^\# & F^\natural
\end{align*}
\]

Set(6-20)

It is obviously dangerous to ascribe too much significance to
such an isolated example in the music of this period. What can
be safely stated is that this passage is a strong indication of
Webern's awareness and use of PC (and of course intervallic)
invariance as a compositional procedure. The extent to which
such procedures are systematically employed remains an un-
resolved question. Similar instances of PC invariance and
transformation appear in this and other movements of Op. 5
(see analysis of movement 4 by Perle, SC & A, 16ff.).

4.2 Op. 5/5

The computer analysis of this movement revealed many signi-
ficant set relations as well as information regarding transposi-
tional choices (to be discussed below). Example 3 shows the
intervallically identical compositional sets. A kind of sym-
metry between the two halves of this piece is reflected in the
utilization of set (4-15), which is first stated as a 4-PC verti-
cality at m. 4, is repeated in m. 5 and then in alternation with
(4-14) throughout the next four measures. Set (4-15) contains
several special properties of particular interest. As indicated
by its vector representation \([1 \ 1 \ 1 \ 1 \ 1 \ 1]\), this set contains an
equal distribution of all six ICs. Furthermore, it is one of
those special sets discussed by Lewin (his category three –
see section 3.8 above), which exists in two intervallically iden-
tical versions which are neither T nor I related. Martino has
included both of these in his table of source tetrachords (which
accounts for his list of 29, as opposed to our 28 4-PC sets).

The vertical statement of (4-14) is generated by equal trans-
position of three of the PCs of (4-15) each down two semitones.
The fourth PC of (4-15) (F$^\#$) is transposed down three semitones
to D$^\#$ rather than E, thus generating a set with non-identical,
but maximally similar interval content (the s.1. = $\sqrt{2}$). The desire to
avoid doubling the E already present in the cello ostinato would
seem a likely motivation for this "unequal transposition" and
the resultant generation of set (4-14). Although this set achieves only local significance (mm. 5-9), set (4-15) is restated in several strategic locations throughout the remainder of the work. Its first recurrence is at the close of the first half of the piece, where it appears as a simultaneity of two full beats duration (at $\frac{1}{4} = 40$) "harmonizing" a melodic statement of (5-7). Most significantly, this statement of (4-15) is Z-related to the initial statements at mm. 4-8. That is, these two statements cannot be related by the operations of T or I. That this indicates Webern's actual awareness of these special set properties is perhaps dubious. What it does strongly suggest is that he was consciously working with sets of intervals rather than (or at least in addition to) sets of pitches, since these Z-related sets may not be "mapped into" each other by the defined operations on pitch.

Set (4-15) returns at m. 17, again in the role of "harmonic accompaniment" to the same "motivic" statement of (5-7). Both sets are stated a major third below their pitch levels in m. 12. Interestingly, the pitch content of set (4-8) which alternates with (4-15) in this passage is identical to that at mm. 10 and 11. (Note that these passages at mm. 10 and 11 are references to the opening of movt. IV, both with respect to interval content of the two lower voices and also with respect to articulation and dynamics.)

In the final statements of (4-15), Webern returns to transposed versions of the initial statements (mm. 4-8). The chord stated in m. 22 is transposed down a perfect 4th, and the spacing is identical to that of the chord in mm. 5-7. Thus the statements of this set throughout the piece are related as follows:

\[
\begin{align*}
\text{mm. 5-8} & \quad \text{Z} & \quad \text{m. 12} & \quad T_4 & \quad \text{mm. 17-18} & \quad \text{Z} & \quad \text{m. 22} \\
\hline
\text{T}_7
\end{align*}
\]

These transpositions reflect a general transpositional scheme which appears to be operative in this piece. Intervally identical sets are related, on both the small and large scales, primarily by $T_7$ ( = $T_5$) and secondarily by $T_4$ ( = $T_8$). On the small scale, the first two statements of (5-18) (m. 3) are $T_4$ related, as are the two successive statements of this vertical set in m. 19. On the large scale it will easily be seen that the analogous statements of the intervally identical sets between mm. 3-4 and mm. 19 are all $T_7$ related. The same $T_7$ relation holds between the statement of (4-8) in mm. 10, 11, (and 17), that in m. 14.
Thus, a kind of symmetrical organization of intervallically identical sets is suggested, in which intervallic analogues occurring in the two halves of the piece are $T_7$ related—or, broadly speaking, the corresponding intervallic sets in the second half of this piece are transposed (registrally, as well as numerically) by perfect fifths. The criteria governing the choice of such "structural" transpositions differ from those of PC invariance found above. It should be noted that pitch class invariance is also employed significantly in this piece. The pitch contents of the last chord in m. 4 (5-16), the return of this set in m. 19, and its horizontalization in m. 20 are all identical, and the "vertical ordering" is actually maintained in the final horizontal statement. The same is true for the succeeding horizontal statements of this set. The viola (m. 20) actually presents the same pitches which comprised the chord on the second beat of m. 4, and the cello (m. 21) presents the PC's of set (4-15) at m. 5. By the addition of the $A\sharp$ which appeared in Vln. I in m. 5., set (5-16) is formed. Note also how Webern links the three horizontal statements of (5-16) in mm. 20-21 by repeating the last PC of the preceding statement as the first PC of the following one.

In addition to pitch class invariance the unification achieved by the consistent employment of identical $T$'s on both "foreground" and "background" levels suggests certain structural procedures analogous to, but by no means identical with, those of tonal music. The problem of determining long range harmonic connections in atonal composition (if, in fact, any actually exist) is obviously crucial to an understanding of structure in these works. It is suggested that by determining intervallic set identities, certain significant "structural analogues" are determined which serve to guide the analyst in the search for these long range connections.

The foregoing analytic examples suggest certain general hypotheses concerning Webern's compositional procedures in his atonal works. These hypotheses are in some measure supported by the statistical analysis which is described below, and therefore a fuller explanation of them is reserved until the concluding section of this paper. It is hoped that the analyses included above have elucidated certain significant aspects of atonal compositional procedure, and have also indicated the value of the analytic approach utilized, both with respect to the defined analytic method and to the employment of the computer in the application of that method.
Interpretation of Output II: Statistical Analysis

5.0 The quantity of data output clearly precluded detailed contextual analysis of the entire sample at this time. It was therefore thought that some basic statistical study of the data might reveal certain general characteristics and techniques in these works. Although statistical analysis was at first applied only to the Webern works, it soon became evident that many of the results of this analysis could not be adequately interpreted without a comparable set of "control" data. Further samples from the atonal works of Schoenberg (Op. 15/1 and Op. 19) and from twelve-tone works by Webern (Op. 23/2) and Schoenberg (4th Quartet) were therefore selected and analyzed.

The only data utilized in the analysis were the interval vectors themselves. They were subject to two types of study: First, frequency counts were made of the compositional occurrences of all interval sets which contained from three to five PCs. Secondly, arithmetic means of the total occurrences of each interval class in these vectors were computed. In this way, the average frequency of each class might, for instance, be compared between individual works and between Webern and Schoenberg in general. Other comparisons made on this basis will be described below. It should be emphasized that our statistical studies were informal and non-rigorous. Standard statistical testing methods such as Student's t-distributions were not employed in evaluating the data. It may be noted that, of course, such tests cannot absolutely validate statistical findings in any event, but only suggest the probability of their accuracy. Thus, at this preliminary stage of investigation, it was felt that musical experience and intuition served as adequate guides to their interpretation.

5.1 Frequencies of intervalllic sets

Frequency counts of compositional sets were restricted to those of 3-, 4-, and 5-PC's, since those of greater magnitude did not occur often enough to supply an adequate sample. An indication of set distribution solely with respect to cardinality is given for Webern's Op. 5 in Table 1. As indicated there, the first and last pieces of this work contained the greatest number of sets, and therefore these two pieces were primarily utilized for the analysis. Frequency counts were also made of set occurrences in the five pieces taken as a whole. These were compared with similar counts taken in the Six Little Piano Pieces, Op. 19 by Schoenberg. The results of these analyses are represented visually (and numerically) in the histograms
of Figures I-XI. Each histogram is arranged so that the total-
ity of possible interval vectors for sets of a certain magnitude
is represented at the bottom of the graph (on the horizontal
axis). Thus, for instance, for sets containing 3 PCs, twelve
positions (corresponding to the twelve 3-PC sets, listed in
Table A, are located on the horizontal axis. The values notated
on the vertical axis represent the number of occurrences of a
set in the given piece. The magnitude of the PC sets repre-
sented on each histogram is indicated in the lower left hand
corner. The total compositional occurrences of the sets of
that magnitude, in the particular composition, as well as the
total number of compositional sets of all magnitudes ("NTOSET")
are indicated on the right-hand margin. In those histograms
which represent "Op. 5 (complete)", the numbers next to the
columns refer to the individual pieces in the opus. For in-
stance, in Figure 1, set (3-3) occurs 37 times in the first piece,
17 times in the third piece, etc.

The following are among the many interesting questions which
these histograms help to illuminate:

1) Which sets of 3-, 4-, and 5-PCs are most frequently
employed by
   a) Webern?
   b) Schoenberg?
   Is this preference consistent in several pieces by each
composer, or is each piece "based on" specific, but
different sets?

2) Are there consistently similar intervalllic character-
istics common to all (or most) such preferred sets?

3) Do the sets used by each composer
   a) in individual pieces
   b) in general
   group closely according to intervalllic similarity?

4) Which sets are never employed by each composer?

5) Which sets are rarely used by each composer, per-
haps used only for special compositional purposes?

6) In general, how restricted is the harmonic vocabulary
   of each composer?

7) Is there a similarity of set distribution in the atonal
   and twelve tone works of each composer?

Detailed descriptions of the results of the statistical analysis
will not be given here, since the histograms are largely self-
 explanatory. Certain of the more significant results will, how-
ever, be briefly indicated below.
## DISTRIBUTION OF SET CARDINALITIES

Op. 5 (Webern)

<table>
<thead>
<tr>
<th>Piece</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-PC sets</td>
<td>22</td>
<td>15</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>54</td>
</tr>
<tr>
<td>3-PC sets</td>
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<td>15</td>
<td>30</td>
<td>7</td>
<td>8</td>
<td>124</td>
</tr>
<tr>
<td>4-PC sets</td>
<td>56</td>
<td>8</td>
<td>7</td>
<td>11</td>
<td>21</td>
<td>103</td>
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<td>10</td>
<td>3</td>
<td>7</td>
<td>3</td>
<td>23</td>
<td>46</td>
</tr>
<tr>
<td>6-PC sets</td>
<td>8</td>
<td>2</td>
<td>8</td>
<td>0</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>7-PC sets</td>
<td>10</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>21</td>
</tr>
<tr>
<td>8-PC sets</td>
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<td>2</td>
<td>0</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>9-PC sets</td>
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<td>0</td>
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<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10-PC sets</td>
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<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ \text{NOTSET} = 180 \quad 44 \quad 63 \quad 30 \quad 72 \quad 389 \]
The tallies revealed an astonishing consistency in Webern’s preference for the 3-PC set (3-3), represented by the interval vector $[1 \ 0 \ 1 \ 1 \ 0 \ 0]$. The predominance of this trichord is a familiar characteristic of Webern’s later twelve tone works, but its extremely predominant use in a work as early as 1909 seemed somewhat surprising. In fact, the general frequency distribution of 3-PC sets in all of the Webern works analyzed (both atonal and twelve-tone) displayed a remarkable consistency. Compare the histograms representing Webern’s 3-PC set usage in Op. 5 complete and in Op. 5/1 (Figures I & III) with Figure VII, showing the frequency distribution of the interval content of disjunct trichords in the 19 twelve-tone sets by Webern.*4 A similar analysis of the second song from Webern’s Op. 23 (a twelve tone work, of course) revealed the same general distribution: Out of the 45 3-PC compositional sets selected, 27 had the interval content $[1 \ 0 \ 1 \ 1 \ 0 \ 0]$. It should be emphasized that compositional sets were selected on the same basis in the twelve tone works as they were in the atonal compositions. Disjunct trichord partitions of the twelve tone set were not considered as a basis for selection. In fact, the interval structure of the twelve-tone set of Op. 23 was not examined until after the compositional set selections were made. This later examination revealed that this set was rather unusual for Webern in that it contained only one minor second in the ordered interval succession, and only one disjunct trichord contained the interval content $[1 \ 0 \ 1 \ 1 \ 0 \ 0]$. The predominance of this trichord in the compositional sets was therefore all the more striking.

In contrast to Webern’s usage, the 3-PC sets in the Schoenberg Op. 19 were far more evenly distributed among all possible sets (see Figure IX). Note that every set is represented at least once. In the excerpts from the openings of the first and second movements of the Fourth Quartet, a predominant usage of set (3-4) and (3-12) was revealed. Of the 86 trichords counted, 36 were examples of the former set, and 15 were examples of the latter.

Concerning excluded 3-PC sets, it is not surprising to find that (3-9), which contains the maximum possible occurrences of IC-5, and (3-10), the “diminished triad”, never appear in the Webern sample. Both appear in the Schoenberg piano pieces.

Note also that of the 53 3-PC sets in the Schoenberg Op. 19, 31 contain no minor 2nds (IC$_1$), whereas in Webern’s Op. 5/1 only 3 such sets occur out of 63. One of the most surprising results of the frequency counts of Webern’s 4-PC sets was the complete
exclusion of set (4-1) (the 4 note chromatic scale segment).

As indicated in Figure II this set does not appear in the entire Op. 5, nor does it occur in Op. 11. Op. 7 shows only two compositional sets with this interval content. The reason for the exclusion of this set by Webern is unclear. A general characteristic of Webern's preferred sets would seem to indicate a tendency towards even intervallc distribution, particularly between IC₁, IC₃ and IC₄. This equal distribution is clearly displayed by set (3-3) which, by virtue of its predominant usage, may be regarded as a kind of intervallc norm in Webern's music. It is of interest to note that each of the four most frequent 4-PC sets in Op. 5 ((4-7), (4-12), (4-18), (4-15)) contains set (3-3).

As noted above, it is a general and consistent tendency of Webern to exclude the sets which contain no IC₁'s, and which emphasize IC₅ and IC₆. Therefore, almost no sets listed below (4-21) or (5-21) appear on the Webern histograms. This trend reflects Webern's bent towards a highly restricted and unified harmonic vocabulary, in which certain types of sets are frequently utilized, others avoided entirely. This results in certain observable "groupings" in the histograms between successive (and therefore intervallcally similar) sets on the horizontal axis. See, for instance, Figures I, III, IV, V, VI. This strongly suggests that intervallc similarity is an important determinant for set selection in Webern's work.

The use of an intervallcally dissimilar set for special purposes may also be revealed by comparing the histograms with the score. Thus in Figure IV we observe that set (4-20) is utilized only once in Op. 5/1. This set has an interval content of [1 0 1 2 2 0], quite different from other sets in the piece. Examination of the score reveals that this set is employed at the climactic point in the piece (see Example 1, m. 50). It is stated fff, as a distinct "harmony" for the motivic statement of set (3-3).

In contrast to the set clusterings exhibited in the histograms of Webern's works, the histograms describing set frequencies in Schoenberg's Op. 19 display a far more equal distribution of almost all possible intervallc sets. Perhaps the most dramatic contrast between the flat appearance of the Schoenberg histograms and the peaked appearance of Webern's may be seen by comparing the distribution of 3-PC sets in each graph. In the excerpts from the Fourth Quartet a general increase in the occurrence of (3-4) and (3-12) was evidenced but this increased
frequency still did not approach the relative frequency of set (3-3) in Webern’s works.

Schoenberg’s characteristic use of dissimilar intervalllic sets is reflected in the output from SUBROUTINE DIFR. In Op. 19, s.i.’s of $\sqrt{8}$ and $\sqrt{10}$ are frequent, in contrast to those in the Webern works, which generally are $\sqrt{4}$ or even $\sqrt{2}$. The distribution of a rather small sample of 4-PC sets in Schoenberg’s Op. 15/1 (Figure XI) reveals a high frequency of two sets, (4-2) and (4-23). Unlike Webern’s frequency groupings, however, these sets are intervallically quite dissimilar—the s.i. equaling $\sqrt{14}$. Furthermore, set (4-2), which emphasizes IC₁ and IC₂ and excludes IC₅ and IC₆ is employed only as a linear set, whereas (4-23), which maximizes IC₅ is used only vertically. This reflects a general characteristic of Schoenberg’s melodic and harmonic usage which will be discussed below. In this respect, Schoenberg differs markedly from Webern, whose general tendency, even in his early works, is to interchange most of his sets freely between the vertical and horizontal planes. This technique has already been observed in the analyses of Op. 5, above, and may also be observed in Op. 7/4.

5.2 Some hypotheses for future study

The marked difference in the contours of the Schoenberg and Webern histograms suggests certain fundamental differences in the “harmonic practice” of the two composers. By severely restricting his intervalllic vocabulary to a few intervalllically similar sets, Webern establishes a means of “harmonic unification” which points to the refinement of the later twelve tone works. The use of such highly similar and limited intervalllic vocabulary largely negates the possibility of achieving harmonic progression (or even distinctions) by means of intervalllic similarity or dissimilarity. As Reti has pointed out (with some anguish) concerning Op. 5, “the interplay between dissonance and consonance...is missing.” (TMM, 94) Therefore, the type of “progression” through pitch class invariance and transformation, discussed in section 4, may be fundamental to Webern’s atonal compositions. The use of PC invariance (largely resulting from set intersections) is a well recognized aspect of Webern’s twelve tone technique, but its application in early atonal music has not been carefully examined. The statistical and contextual analysis of intervalllic sets in Webern’s Op. 5 (as well as in Op. 7 and Op. 11) suggests that certain recurrent intervalllic sets may represent “harmonic” norms. Certainly the frequency of set (3-3) in both early and late works would suggest its use as such a norm. Another
frequently used set is (4-7) which, in addition to its employment in Op. 5 is also significant in Op. 7 and in several twelve tone works examined.

Through the recurrence of these (intervallically) identical sets, and the higher degree of similarity between all of the sets employed, Webern achieves a kind of equalization of intervalllic ("harmonic") content. This equalized intervalllic situation necessitates the use of other compositional methods for achieving structural differentiation. The similarity of and dissimilarity of pitch content appears to be an important means of differentiating the harmonic sets and creating a sense of "harmonic progression" between them. The increased importance of durational considerations and the structural use of such compositional aspects as register, timbre and dynamics is also better understood in this context.

5.3 Determination of intervalllic means; further hypotheses

In addition to making frequency counts of sets of intervals, it was thought of interest to calculate the frequencies of each interval class within these sets. In a given work, the average frequency of occurrence of each IC in compositional sets of each magnitude could then be computed in the following way: Take the vectors which represent the interval content of each of the compositional sets of n-PCs. Sum the number of occurrences of each IC in all the vectors. Then divide by the total number of vectors to determine a "mean interval vector". The following example should make this clear:

\[
\begin{align*}
\text{IVEC}_1 & \quad [3 \ 2 \ 1 \ 0 \ 0 \ 0] \\
\text{IVEC}_2 & \quad [3 \ 2 \ 1 \ 0 \ 0 \ 0] \\
\text{IVEC}_3 & \quad [1 \ 1 \ 1 \ 1 \ 1 \ 1] \\
\text{total} & = 7 \ 5 \ 3 \ 1 \ 1 \ 1 \\
\text{mean} & \quad \frac{[7 \ 5 \ 3 \ 1 \ 1 \ 1]}{3} = [2.34 \ 1.67 \ 1.0 \ .33 \ .33 \ .33]
\end{align*}
\]

Thus in our hypothetical "piece" composed of 3 compositional sets, the average number of occurrences of IC\textsubscript{1} equals 2.34, compared to an average of 1.00 IC\textsubscript{3}’s. Clearly these figures are not meaningful without some further standard of reference. In addition, the distributional characteristics of all the possible sets of each magnitude make certain IC’s more or less probable. It was noted in section 2.0, for instance, that IC\textsubscript{6}
tended to occur approximately half as many times as the other five ICs.

It was therefore thought that these problems could be largely compensated for by computing a mean of all vectors for each set of sets of each magnitude, which could then be used as a standard of reference. This "standard mean vector" may be regarded as exhibiting the frequency of IC occurrence which would result from a large random selection of n-PC sets, where occurrence of each set of the total population was equiprobable. By comparing the mean vector of compositional sets of the same magnitude, deviations are computed which reflect the constraints upon each IC that distinguish it from a random selection. Thus, in our example above, the deviations from the standard (random) mean are calculated as follows:

<table>
<thead>
<tr>
<th>compositional mean</th>
<th>[2.34 1.67 1.00 .33 .33 .33]</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard mean</td>
<td>[1.07 1.07 1.11 1.07 1.07 .61]</td>
</tr>
<tr>
<td>deviations of compositional IC distribution</td>
<td>+1.27 +.60 -.11 -.74 -.74 -.28</td>
</tr>
<tr>
<td>from standard</td>
<td></td>
</tr>
</tbody>
</table>

Through the use of this standard vector, two or more vectors representing the "mean interval content" of different works (or different aspects of the same work) may be compared. The graphs in Figures XII-XVI represent these deviations visually. They display three types of comparisons:

1 Intervalle distribution in atonal works of
   a) Webern
   b) Schoenberg

2 Emphasis of each IC in vertical as opposed to horizontal sets by each composer.

3 Distributional similarity of the six ICs in atonal, and selected 12-tone works by each composer.

The six equally spaced positions on the horizontal axis represent the six ICs. The numerical values plotted on the vertical plane indicate the degree of deviation from the standard mean vector. Thus, for instance, the graph of deviations for 3-PC sets in Figure XII indicates that Webern's Op. 5 exhibits a frequency of IC1 that is .43 greater than that which would result from a random selection of sets of this magnitude. In contrast to this, the frequency of IC5 is .27 less than that same standard of reference. Since this graph obviously represents the frequencies of discrete values (i.e., the six ICs), only locations
of the points themselves indicate the values of the deviations, and the connective lines serve only as a visual aid. Their use in illustrating distributional contour is immediately apparent upon examination.

Several of these graphs illustrate findings that were anticipated by previous experience with these compositions. The findings mentioned above, concerning Webern's emphasis of IC₁ and de-emphasis of IC₅ is one such example. That these graphs indicated intervallic characteristics which might have been expected offered strong support for the validity and accuracy of our methodology. Without detailing the findings represented on the graphs, certain significant trends may be mentioned.

The contours of all the graphs describing Webern's intervallic usage show a remarkable consistency. In all eight representations of Webern's works, the general contour indicates the strongest emphasis of IC₁, balanced by the strongest de-emphasis of IC₂. The relation between IC₃ and IC₄ seems somewhat variable, but both generally fall in the range from 0 to +0.3. IC₅ is always de-emphasized, falling in the range between -0.2 and -0.4. Tritones (IC₆) exhibit no strong tendency in either direction, though they generally tend to be somewhat de-emphasized.

Remarkably, the contour of the graphs of Webern's Op. 5 is almost exactly paralleled in the graphs which describe Op. 23/2. See the first two examples in Figure XII.

The Schoenberg example of 3-PC sets from Op. 19 which is here plotted against the Webern examples displays a contour exactly opposite to the Webern. IC₁, IC₃, and IC₆ are de-emphasized, and IC₂, IC₄, and (especially) IC₅ all fall above the 0 axis. It may be noticed that the deviations in the Schoenberg example are not nearly so marked as in the Webern, and also that his graphs do not show the same consistency of contour as those of the Webern works. This "leveling" effect is similar to that noted in the histograms, and is a reflection of Schoenberg's more diverse intervallic vocabulary in contrast to the highly constrained usage of Webern.

It should be noted that Figure XVI, which compares sets from Op. 19 and Op. 37 of Schoenberg, also reveals a general similarity of contour between the representation of the atonal and twelve-tone works, although (as would be expected) the twelve-tone work exhibits a more highly constrained intervallic vocabulary.
Finally, it was thought to be of interest to compare intervallic usage between the vertical and horizontal compositional sets within the same piece. The results proved most illuminating. A glance at the first graph of Figure XIII reveals the striking contrast in Schoenberg's vertical and horizontal intervallic usage of 3-PC sets.

The contours shown there indicate Schoenberg's strong predilection for IC₁ and IC₂ in his horizontal set statements, and a strong emphasis on IC₅ (and IC₄) in the verticalities. These general contours are followed (with some variations) in the 4-PC and 5-PC set graphs in the figure. Figure XIV shows the extraordinary emphasis of IC₅ (and corresponding de-emphasis of IC₁) in the vertical compositional sets of Op. 15/1. These tendencies are exactly those which one would expect in tonal compositions—that is, a strong emphasis on the "melodic" intervals (major and minor 2nds) in the horizontal plane, and the predominance of the "harmonic" 5ths (and major and minor 3rds) in the verticalities. This distribution would seem to suggest that Schoenberg was still influenced by certain fundamental aspects of the tonal system.

Webern's equating of the vertical and horizontal is a frequently discussed aspect of his later style; our study reveals a significant tendency towards this as early as 1910. Figure XV reveals a similar distributitional contour for both vertically and horizontally stated intervals. Thus, the findings seem to suggest that as early as 1910, the pupil Webern was tending towards a more radical harmonic practice than his master.

Conclusion

This study was directed toward the analysis of intervallic relations in atonal music. To this end an analytic method was formulated and tested. The application of the method was made feasible through the use of a high-speed digital computer. Therefore input and output operations as well as the central core of the work, the program with its specially designed subroutines, were described in detail.

Interpretation of the output posed most interesting questions, it is believed, and an effort has been made to elucidate these questions, to suggest answers, and in some cases, to indicate ways in which answers might be discovered. It is hoped that results of a more general nature will be obtained in subsequent studies, for which the present one provides a broad base.
WEBERN OP.5/1 (COMPLETE)
TOTAL 3-PC SETS = 121
NTOSET = 389
WEBERN OP. 5/1
TOTAL 4-PC SETS = 106
WEBERN OP. 5/1
TOTAL 3-PC SETS = 63
NTOSET = 180
WEBERN OP, 5/1
TOTAL 4-PC SETS = 56
NTOSET = 180
WEBERN OP. 5/5
TOTAL 4-PC SETS = 21
NTOSET = 72

TOTAL 5-PC SETS = 23
WEBERN 12-TONE SETS
INTERVAL CONTENT OF DISJUNCT TRICHRDS
TOTAL 3-PC SETS = 76
ARNOLD SCHOENBERG OP. 19 (COMPLETE)

TOTAL 4-PC SETS = 75
NTOSSET = 201

TOTAL 3-PC SETS = 53

TOTAL 5-PC SETS = 31
ARNOLD SCHOENBERG OP. 15/1
TOTAL 4-PC SETS = 27
figure 12

DEVIATION FROM STANDARD MEAN VECTORS
WEBERN OP. 5 (COMPLETE) REPRESENTED BY
SCHOENBERG OP. 19 REPRESENTED BY
SCHOENBERG OP. 23/2 REPRESENTED BY ---
DEVIATIONS FROM STANDARD MEAN VECTORS
SCHOENBERG OP.19
VERTICAL SETS REPRESENTED BY - - - HORIZONTAL SETS BY —
DEVIATIONS FROM STANDARD MEAN VECTORS
SCHOENBERG OP.15/1
VERTICAL SETS REPRESENTED BY - - HORIZONTAL SETS BY ---

4-NOTE SETS
figure 15

DEVIATIONS FROM STANDARD MEAN VECTORS
WEBERN OP. 5
VERTICAL SETS REPRESENTED BY — — HORIZONTAL SETS BY —
DEVIATIONS FROM STANDARD MEAN VECTORS
SCHOENBERG OP.19 AND 37
NOTES

1 For a description of a recent effort in this direction see Howe and Jones.

2 See Forte, JMT.

3 I am greatly indebted to Professor Allen Forte for this tabulation, and for the terminology which describes its organization. I am also indebted to him for an understanding of many of the properties of and relations between both the PC and IC sets. Such an understanding certainly suggested many of the procedures described in this paper.

4 I would like to thank Mr. John Rothgeb for the use of his paper, "Anton Webern's Twelve-Tone Sets", which contains a tabulation of the interval content of all the disjunct trichords of the Webern sets.

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