All Possible Scales
(Optimal Well Temperament)

Larry Polansky

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with Dan Rockmore, Kimo Johnson and douglas repetto

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Part 1: The "Historical Tuning Problem"

For two primes p and q, there are no integers m, n, such that:

$$p^m = q^n$$

In other words, what is called the Diophantine equation,

$$3^m = 2^n$$

has no solutions.

This means, informally, that *any* tuning system which more than one prime will at some point be "out of tune" with itself. Further, since primes are the "building blocks" of all integers, this applies to any whole number based tuning system.

No scale can include two primes and "resolve." That is, any a scale based on rational numbers must, in some sense (that needs defining), be *inconsistent*.

This fact has been crucial in the development of tuning systems throughout history, and around the world.

Simple Example: Pythagorean System

The most well-known and simplest demonstration of what might be called *prime collision* can be seen in a scale built on the simplest non-octave ratio, 3:1.

Since $3^m \neq 2^n$ (always), no cycle of fifths will ever "resolve" to an octave (*2:1*).

Note that taking the $\log_2$ of both sides gives

$$m/n = log_2 3$$

— which, using simple mathematical techniques, yields a number of rational number approximations [1].

Constraints
Commonly, scales around the world and throughout history have had at least two constraints: *repeat factor* and *number of degrees*.

Repeat Factor
Many scales have, as a constraint, some notion of *repetition*. This is usually the octave (*2:1*), but does not have to be. Scales have been hypothesized which repeat at the *3:2*, or any other ratio. Other scales, like central Javanese slendro, use repeat factors which "stretch" the octave by some small amount (perhaps *10-20¢*). Repeat factor may be seen as a kind of special case of prime collision (if one scale value is fixed in this way, all other intervals must "deal" with it somehow).

Number of Degrees
Most scales are fixed, or at least limited, to a small number of pitches, or scale degrees. This is usually motivated by physical instruments which have some number of keys, strings, holes, etc.

Note that without these two constraints, prime number collision is less of a problem, if at all.

Keeping the wolf from the door: compromise and scale

The wolf-fifth (on the supertonic) in classic just intonation

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1/1    9/8     5/4       4/3    3/2    5/3     15/8    2/1
   9/8    10/9    16/15    9/8   10/9    9/8    16/15
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The interval between the 2^{nd} degree (9/8) and the 6^{th} degree (5/3) is 40/27, or the "wolf 5^{th}". Changing the 6^{th} degree to 27/16 makes the 5^{th} build upon the second degree "pure," but makes the 5^{th} built upon the 6^{th} degree the wolf.

The usual example of "prime collision" in just intonation involves the tuning of scale degrees 2, 3, and 6. There is a "good" 5th between 2 to 6, and 6 up to 3. However, the 5th between the 2nd and 6th scale degrees is, historically, not anyone's idea of an "ideal" ratio.

This shows the conflict between 3- and 5-limit tuning systems, and could be said to have been the motivation for the long history of "compromise" tunings (mean-tones, well-temperaments, and finally, 12-ET).

There are, of course, many, in fact infinitely many, other possible instances of this general problem (I call this the *canidae* problem, it's not the genus, but the "family"). The problem is the "collision" that must occur in scale systems which use more than one prime, not just 3- and 5-: even 3- and 2- cause this problem. (In a great deal of contemporary music this is a feature, not a bug)

A detailed "cookbook" explanation of the wolf-5th:

If the 2nd of a simple diatonic just scale is tuned as 9/8, and the 6th as 5/3, then the interval between those two scale degrees (a P5th) becomes, 5/3 / 9/8 = 40/27 (-20¢ of a 3/2): the wolf-fifth.

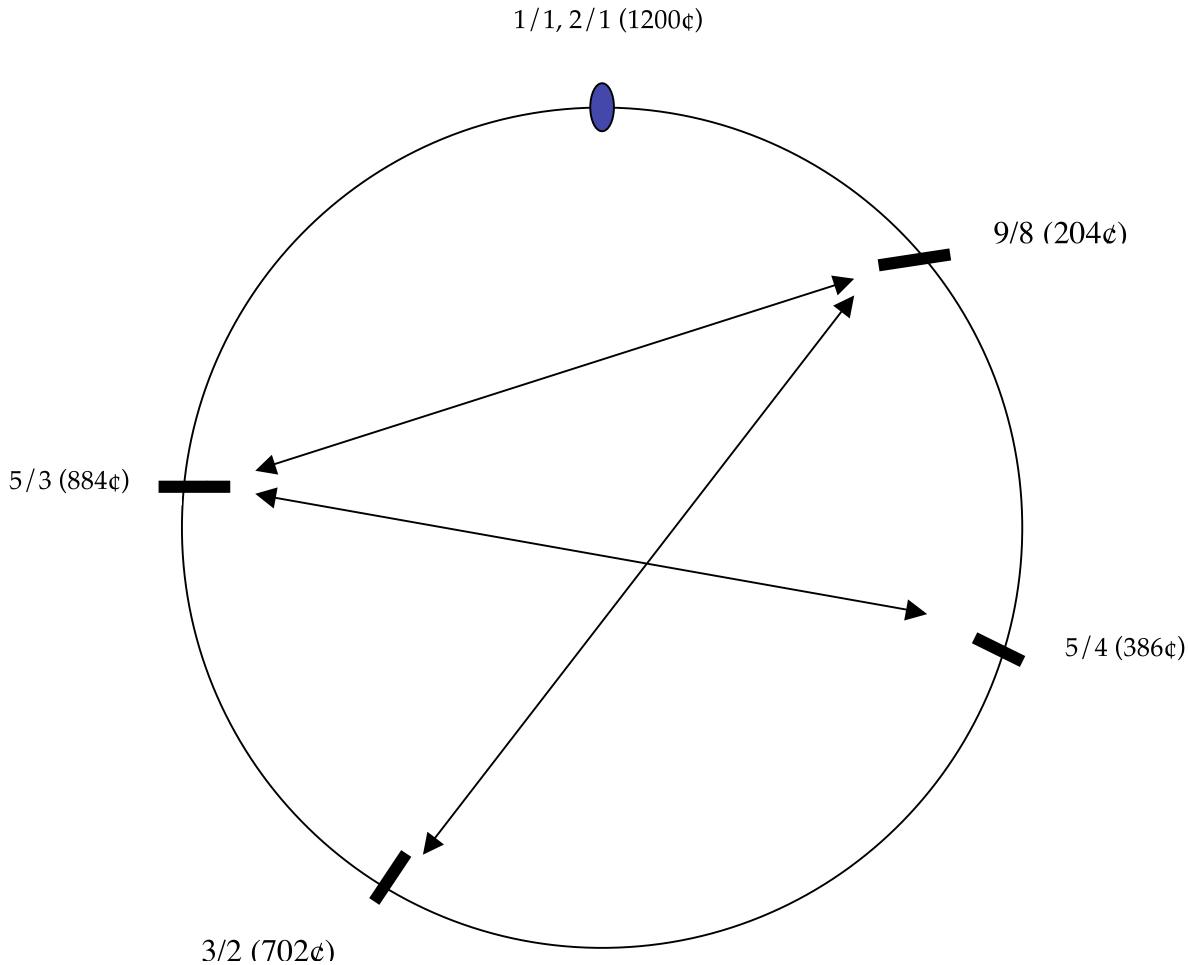
That P5th (between the 2nd and 6th degrees), necessary, for example, in the supertonic triad, will be "out of tune."

However, the 5th from 5/3 up to 5/4 remains "ideal" (3/2).

Alternatively, if the 6th degree is tuned as a Pythagorean 27/16, and the 3rd as (a just) 5/4, then the same situation arises as a P5th upon the submediant (27/16 / 5/4 = 27/20, the P4th, or is the inversion of 40/27).

Just Intonation as a Circle

Note that it is not just the 3-5 collision that causes the wolf-fifth, but the repeat factor (2/1). The 6th degree *up to* the 3rd degree is usually cited as the problem (not the inversion, the 6th degree down to the 3rd degree). In other words, the scale "wraps" around some fixed point (2/1). All other distances between intervals must be "juggled" over the circumference of the circle. The 2/1 can't move. But, when one interval is moved, *all* intervals are changed.



Mean-tone, Well-temperament

Historically, *mean-tone temperaments* preserve just relationships exactly for a few intervals, and compromise others by a fixed amount (1/4-comma, 1/6-comma, etc). Mean-tones tend to be very good (in approximating important ideal intervals) in a few (important) keys, and quite bad, or, by definition, useless, in "outer" keys.

Well-temperaments usually have few (if any) exact ideal intervals — the "error" is distributed widely over the entire scale. *Ideal* intervals (often, historically, 5/4 3rds and 3/2 5ths) are *approximated* to most scale degrees. Few or none of the intervals are exact. These tunings may be thought of as techniques for *fitting* a fixed number of pitches to a fixed set of tuning relationships (or constraints).

Consequently, well-temperaments are thought to be "more in tune" for a greater number of musical keys than MTs, which are usually "good" for some keys, and "not so good" for others. WTs do preserve the notion of "good" and "less good" keys, as well as "good" and "less good" intervals, but they tend to be more complex than MTs, and fit more intervals and more keys better.

WTs are perhaps the most interesting set of tuning systems in that they approximate some set of *ideal* intervals (like just ratios) in the context of a fixed set of pitches, keys, etc. Although the term has a more specific historical meaning in western art music, a general interpretation of the term could include MTs, Central Javanese slendro and pelog, many other world music systems, and even equal temperament.

An example of an historical well-temperment (Werckmeister III)

This tuning is often thought to be one of the major advances in historical tunings, and possibly the one intended by Bach in the *WTC* (or maybe one of Kirnberger's tunings). W3 is (rounded to the nearest cent):

C#	**90¢**
D	**192¢**
Eb	**294¢**
E	**390¢**
F	**498¢**
F#	**588¢**
G	**696¢**
G#	**792¢**
A	**888¢**
Bb	**996¢**
B	**1092¢**
C	**1200¢**

Rasch's analysis of "Werckmeister III" [2] is a good example of how well-temperaments work (historically).

There are four tempered 5ths (each flat by about 6¢), three of which lie in the "center" of the tuning (G/C, D/G, A/D). All other fifths are pure (3/2). For example, Bb/Eb = C#'/F# = 702¢. None of the thirds are "pure" (5/4). The "central" ones are "rather good" but too wide (e.g E/C = 390¢), and the "peripheral ones" (from the key of C around the circle of fifths) Pythagorean (Bb/F# = 406¢, roughly equivalent to the 81/64 Pythagorean 3rd, about 21¢ wide of the 5/4 just).

> "In such a tuning the central tonalities (with only a few or no sharps or flats) are rather good. The peripheral tonalities (with many sharps or flats) are not too bad, and tolerable, in any case. There is no clear distinction betwen ordinary and wolf intervals, like there is in mean-tone tuning. There are no wolves. All tones and intervals can be used enharmonically... With this tuning, Werckmeister has fulfilled his own demands to construct a tuning in which all tonalities could be performed without... the disturbing effects of wolf intervals"

Note that the evaluation is based on 3rds and 5ths (*intervals*), as well as "centrality" (*key*).

Scale Evaluation Functions

Based on how well the tempered intervals match ideal (just) 5^{th}s and 3^{rd}s, Rasch develops a number of measures for WTs:

1) the "mean tempering of a major triad ... the mean of the absolute values (in cents) of the temperings of the $P5^{th}$, the $M3^{rd}$, and the $m3^{rd}$ in the triad"
2) the "mean tempering of a key... the weighted mean of the temperings of all triads" (triads are given successively lower weights starting from the tonic, and proceeding downward through the circle of fifths)
3) the "mean tempering of a tuning... the mean tempering of all consonant intervals, ...equal to the mean tempering of all triads, or of all keys"

Chalmers [3] uses the idea of "error" in his evaluation and creation of what he refers to as "linear temperaments" by the *Method of Least Squares*. Beginning with the just $M3^{rd}$ and $P5^{th}$, he extends his method to other ratios, finding new tunings which are "optimized" MTs (MTs with minimized errors). Chalmers also points out that the design of the error function is independent of the particular ratios desired. "There are innumerable ways in which the error functions can be combined. Various means, arithmetic, harmonic, geometric, to name the simplest, may be used."

Both Chalmers and Rasch thus consider various temperaments as approximations, in some way, of ideal tunings. The error calculation (whatever that might specifically mean) is a first step towards designing a tuning which optimally fits some *set of ratios*, *weights*, and *predefined error function*.

Werckmeister III as a half-matrix

One way to look at a scale is as a half-matrix of relationships. For n pitches, there are

$$(n^2 - n)/2$$

possible intervals contained in the scale (the second order binomial coefficient, or the "half-matrix minus the diagonal").

This is almost equivalent to the circle representation of just intonation shown above, except that instead of connecting arrows between pitches (whose length corresponded to interval size), the size of the interval is written in the appropriate position in a pairwise relationship matrix.

W3 has 12 m2[nd]s, 11 M2[nd]s, 10 m3[rd]s, and so on. Each diagonal represents one interval (this particular geometry is arbitary, intervals are given in cents).

	C	**C#**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**C**
C		90	192	294	390	498	588	696	792	888	996	1092	1200
C#			102	204	300	408	498	606	702	798	906	1002	1110
D				102	198	306	396	504	600	696	804	900	1008
Eb					96	204	294	402	498	594	702	798	906
E						108	198	306	402	498	606	702	810
F							90	198	294	390	498	594	702
F#								108	204	300	408	504	612
G									96	192	300	396	504
G#										96	204	300	408
A											108	204	312
Bb												96	204
B													108

(The ½ matrix of W3)

What we would like (all good 3rds, good 5ths, good 4ths, but not possible!) (*DC = don't care!)*:

	C	**C#**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**C**
C		DC	DC	DC	**386**	**498**	DC	**702**	DC	DC	DC	DC	**1200**
C#			DC	DC	DC	**386**	**498**	DC	**702**	DC	DC	DC	DC
D				DC	DC	DC	**386**	**498**	DC	**702**	DC	DC	DC
Eb					DC	DC	DC	**386**	**498**	DC	**702**	DC	DC
E						DC	DC	DC	**386**	**498**	DC	**702**	DC
F							DC	DC	DC	**386**	**498**	DC	**702**
F#								DC	DC	DC	**386**	**498**	DC
G									DC	DC	DC	**386**	**504**
G#										DC	DC	DC	**386**
A											DC	DC	DC
Bb												DC	DC
B													DC

(An "ideal" W3 ½ matrix)

From these representations of WT as a matrix (which is similar to the circle used for just intonation above), we can compute an *error matrix*, which tells us how far all actual intervals are from some notion of ideal interval. In the example below, just to simplify things, only the M2nds, M3rds, P4ths, and P5ths are considered. The entries in each column are the absolute error (in cents) for each occurrence of those intervals (where the ideal intervals, respectively, are 9/8, 5/4, 4/3, and 3/2).

Error Matrix (four intervals only) for WIII

	C	**C#**	**D**	**Eb**	**E**	**F**	**F#**	**G**	**Ab**	**A**	**Bb**	**B**	**C**
C		—	12	—	6	0	—	6	—	—	—	—	1200
C#			—	0	—	24	0	—	0	—	—	—	—
D				—	6	—	12	6	—	6	—	—	—
Eb					—	0	—	18	0	—	0	—	—
E						—	6	—	18	0	—	0	—
F							—	6	—	6	0	—	0
F#								—	0	—	24	6	—
G									—	12	—	12	6
G#										—	0	—	24
A											—	0	—
Bb												—	0
B													—

The matrix shows how important the P5th is in W3 are (and the inversion), but that the M3rds tend to be a bit wider than just (in fact, they are closer to Pythagorean). But more importantly, perhaps, this *error matrix* suggests ways to evaluate a scale to some *explicit criteria*, and further, to create scales to conform to them.

The sum of all the entries in this error matrix is essentially a measure of how well a scale fits some ideal set of criteria for that scale. In this case, those criteria are just four *intervals* and the *repeat factor*, with no *weighting* as to the importance of any of those intervals, nor to the *keys* in which they occur. By *minimizing* that error function, we can *optimize* that scale.

Part 2: Optimal Well-Temperament

Statement of the problem as an error function on a constrained system

The problem of developing a scale might thus be defined as the *optimal* solution to the following set of constraints:

- some *repeat factor* ω (e.g., an octave)
- some number of "desired" or *ideal ratios* $\mathbf{I_1}$**,** … $\mathbf{I_n}$ (= (number of pitches) -1) for intervals (that is, all intervals between scale pitches)
- some notion of relative importance for these intervals $\mathbf{i_1}$**,** … $\mathbf{i_n}$ (*interval weights*)
- some notion of relative importance for "keys," $\mathbf{k_1}$**,** … $\mathbf{k_n}$ or the fixed pitches in the scale to which intervals are measured (*key weights*)
- some fixed *number of pitches* **n + 1** (e.g., a scale, which has one more pitches than the number of intervals)

The *error function E* for a scale is a measure of how closely a given scale fits the *definition* of an ideal scale specified by the above criteria.

In simpler terms, we have a certain *number of pitches* (for example, we're tuning a keyboard, or a génder); a few (abstract) *intervals* we consider important (P5th, M3rd, etc.). We also want certain "keys" or "pathets" or "tonal centers" or "whatever" to be in tune at the expense of other, less important "keys" (as we've seen, we can't have it both ways). We want that scale to repeat at some interval (say, an octave).

What is the best scale that we can come up with that **optimizes** its total network of intervallic relationships given that criteria? That scale, by defintion, will be the "best" scale for the given criteria.