The tuning problem
Statement: Given

- the **octave** ω
- the **number of intervals** n
- the **intervals** $I_1, \dots, I_n$
- the **interval weights** $\iota_1, \dots, \iota_n$
- the **key weights** $\kappa_0, \dots, \kappa_n$ (note one more than number of intervals)

The “matrix” M: We let $a_1, \dots, a_n$ denote the keys that we are looking for, i.e., those frequency values that will make up an octave

$$0, a_1, \dots, a_n, \omega$$

give M, an n by $n+1$ matrix defined by

$$M = \begin{pmatrix} a_1 - 0 & a_2 - a_1 & a_3 - a_2 & \dots & a_n - a_{n-1} & \omega - a_n \\ a_2 - 0 & a_3 - a_1 & a_4 - a_2 & \dots & \omega - a_{n-1} & \omega + a_1 - a_n \\ a_3 - 0 & a_4 - a_1 & a_5 - a_2 & \dots & (\omega + a_1) - a_{n-1} & (\omega + a_2) - a_n \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ a_n - 0 & \omega - a_1 & (\omega + a_1) - a_2 & \dots & (\omega + a_{n-2}) - a_{n-1} & (\omega + a_{n-1}) - a_n \end{pmatrix}$$

Thus, thinking of the keys as ”wrapped on the circle” (LP - there is a better way to say this, but I have to think about it) row 1 gives all the intervals from one key to the next, row 2 is all intervals of length 2, and so on.

The matrix M simply organizes the intervals as indicated above. We are interested in an associated error function $E = E(a_1, \dots, a_n)$ which measures the sum of squares of differences of each of the intervals from their ideal difference - these are the I_j:

$$E = [(m_{11} - I_1)^2 + \dots + (m_{1,n+1} - I_1)^2 + \dots + (m_{jk} - I_j)^2 + \dots + (m_{n,n+1} - I_n)^2]$$

This is the *unweighted* version of the error function. The *weighted* version is

$$E = [\iota_i(\kappa_0 m_{11} - I_1)^2 + \dots + \iota_1(\kappa_n m_{1,n+1} - I_1)^2 + \dots + \iota_j(\kappa_{k-1} m_{jk} - I_j)^2 + \dots + \iota_n(\kappa_n m_{n,n+1} - I_n)^2]$$

The problem of minimizing this can be done using the tools of multi-variable calculus. This is a quadratic function of $a_1, \ldots, a_n$. In one variable, a quadratic function is some kind of parabola. In order to find an extreme value, either a min or a max, the tools of calculus tell us - in one variable - to take a derivative, set the resulting function (the first derivative) equal to zero and then solve. At that value the original function takes on an extreme value (usually - there are other possibilities, but let's forget that for now).

In several variables all the same ideas apply. However, in this case we need to take first *partial derivatives* with respect to each of the variables $a_1, \ldots, a_n$, leading us to look at the set of simultaneous linear equations:

$$\begin{aligned} \frac{\partial E}{\partial a_1} &= 0 \\ \frac{\partial E}{\partial a_2} &= 0 \\ &\vdots \\ \frac{\partial E}{\partial a_n} &= 0 \end{aligned}$$

Which, assuming that things aren't too bad, we solve. Note that it is n equations in n unknowns.

Specific example:

- the **octave** $\omega = 1200$
- the **number of intervals** $n = 2$
- the **intervals** $I_1 = 400, I_2 = 800$
- the **interval weights** $\iota_1 = 1, \iota_2 = 1$
- the **key weights** $\kappa_0 = 1, \kappa_1 = 1, \kappa_2 = .2$ (note one more than number of intervals)

$$M = \begin{pmatrix} a_1 & a_2 - a_1 & 1200 - a_2 \\ a_2 & 1200 - a_1 & (1200 + a_1) - a_2 \end{pmatrix}$$

So that the unweighted error function is

$$\begin{aligned} E_{unw} &= [400 - a_1]^2 + [400 - (a_2 - a_1)]^2 + [400 - (1200 - a_2)]^2 + \\ &\quad [800 - a_2]^2 + [800 - (1200 - a_1)]^2 + [800 - ((1200 + a_1) - a_2)]^2 \\ &= [400 - a_1]^2 + [400 - a_2 + a_1]^2 + [a_2 - 800]^2 + \\ &\quad [800 - a_2]^2 + [a_1 - 400]^2 + [a_2 - a_1 - 400]^2 \end{aligned}$$

and the weighted version is

$$\begin{aligned}
E_w &= [400 - a_1]^2 + [400 - (a_2 - a_1)]^2 + [400 - .2(1200 - a_2)]^2 + \\
&\quad [800 - a_2]^2 + [800 - (1200 - a_1)]^2 + [800 - .2((1200 + a_1) - a_2)]^2 \\
&= [400 - a_1]^2 + [400 - a_2 + a_1]^2 + [400 - .2(1200 - a_2)]^2 + \\
&\quad [800 - a_2]^2 + [a_1 - 400]^2 + [800 - .2((1200 + a_1) - a_2)]^2
\end{aligned}$$

Now we take partials with respect to a_1:

$$\begin{aligned}
\frac{\partial E_w}{\partial a_1} &= -2[400 - a_1] + 2[400 - a_2 + a_1] + 2[a_1 - 400] + 2[800 - .2((1200 + a_1) - a_2)] * .2 \\
&= -2[400 - a_1] + 2[400 - a_2 + a_1] + 2[a_1 - 400] + .4[800 - .2((1200 + a_1) - a_2)]
\end{aligned}$$

and then a_2:

$$\begin{aligned}
\frac{\partial E_w}{\partial a_2} &= -2[400 - a_2 + a_1] + 2[400 - .2(1200 - a_2)] * (-.2) \\
&\quad -2[800 - a_2] + 2[800 - .2((1200 + a_1) - a_2)] * .2 \\
&= -2[400 - a_2 + a_1] - .4[400 - .2(1200 - a_2)] - 2[800 - a_2] + \\
&\quad .4[800 - .2((1200 + a_1) - a_2)]
\end{aligned}$$

and if we set these equal to zero we get a solution of

$$a_1 = 412.698, \qquad a_2 = 714.042$$