5 Classification, characterization, and analysis of tetrachords

THIS CHAPTER CONTAINS a complex mixture of topics regarding the description or characterization of tetrachords. Some of the concepts are chiefly applicable to single tetrachords, while others refer to pairs of tetrachords or the complete tetrachordal space. The most interesting of the newer methods, those of Rothenberg and Polansky, are most usefully applied to the scales and scale-like aggregates described in detail in chapter 6. Moreover, Polansky's methods may be applied to parameters other than pitch height. The application of these techniques to tetrachords may serve as an model for their use in broader areas of experimental intonation.

The first part of the chapter is concerned with the historical approach to classification and with two analyses based on traditional concepts. These concepts include classification by the size of the largest, and usually uppermost, incomposite interval and subclassification by the relative sizes of the two smallest intervals. A new and somewhat more refined classification scheme based on these historical concepts is proposed at the end of this section.

These concepts and relationships are displayed graphically in order that they may become more intuitively understood. A thorough understanding of the melodic properties of tetrachords is a prerequsite for effective composition with tetrachordally derived scales. Of particular interest are those tetrachords which lie near the border of two categories. Depending upon their treatment, they may be perceived as belonging to either the diatonic or chromatic genera, or, in other cases depending on the CIs, to either the enharmonic or chromatic. An example is the intense chromatic or soft diatonic types, where the interval near 250 cents may be perceived as either a large whole tone or a small minor third. This type of ambiguity may be made compositionally significant in a piece employing many different tetrachords.

The middle portion of the chapter deals with various types of harmonic and melodic distance functions between tetrachords having different intervals or intervallic arrangements. Included in this section is a discussion of the statistical properties of tetrachords, including various means (geometric mean, harmonic mean, and root mean square; see chapter 4) and statistical measures of central tendency (mean deviation, standard deviation, and variance). Both tabular and graphical representations are used; the tabular is useful to produce a feeling for the actual values of the parameters.

These concepts should be helpful in organizing modulations between various tetrachords and tetrachordal scales. For example, one could cut the solid figures generated by the various means over the whole tetrachordal space by various planes at different angles to the axes. The intersections of the surfaces with the planes or the interiors of the bounded portions of the figures of intersection define sets of tetrachords. Planes parallel to the bases define tetrachordal sets with invariant values of the means, and oblique planes describe sets with limited parametric ranges. Similarly, lines (geodesics) on the surfaces of the statistical measures delineate other tetrachordal sets. These techniques are similar to that employed by Thomas Miley in his compositions *Z-View* and *Distance Music*, in which the intersections of spheres and planes defined sets of intervals (Miley 1989).

The distance functions are likewise pertinent both to manual and algorithmic composition. James Tenney has used harmonic and melodic distance functions in *Changes: Sixty-four Studies for Six Harps*, a cycle of pieces in 11-limit just intonation. Polansky's morphological metrics are among the most powerful of the distance functions. Polansky has used morphological metrics in a number of recent compositions, although he has not yet applied them to sets of tunings (Polansky, 1991, personal communication). His compositions employing morphological metrics to date are *17 Simple Melodies of the Same Length* (1987), *Distance Musics I-VI* (1987), *Duet* (1989), *Three Studies* (1989) and *Bedhaya Sadra/Bedhaya Guthrie* (1988–1991).

In the absence of any published measurements known to the author of the perceptual differences between tetrachordal genera and tetrachordal permutations, the question of which of the distance functions better models perception is unanswerable. There may be a number of interesting research problems in the psychology of music in this area.

The chapter concludes with a discussion Rothenberg's concept of *propriety* as it applies to tetrachords and heptatonic scales derived from tetrachords. Rothenberg has used propriety and other concepts derived from his theoretical work on perception in his own compositions, i.e., *Inharmonic Figurations* (Reinhard 1987).

Historical classification

The ancient Greek theorists classified tetrachords into three genera according to the position of the third note from the bottom. This note was called *lichanos* ("indicator") in the hypaton and meson tetrachords and *paranete* in the diezeugmenon, hyperbolaion, and synemmenon tetrachords (chapter 6). The interval made by this note and the uppermost tone of the tetrachord may be called the *characteristic interval* (CI), as its width defines the genus, though actually it has no historical name. If the lichanos was a semitone from the lowest note, making the CI a major third with the 4/3, the genus was termed enharmonic. A lichanos roughly a whole tone from the 1/1 produced a minor third CI and created a chromatic genus. Finally, a lichanos a minor third from the bottom and a whole tone from the top defined a diatonic tetrachord.

The Islamic theorists (e.g., Safiyu-d-Din, 1276; see D'Erlanger 1938) modified this classification so that it comprised only two main categories translatable as "soft" and "firm." (D'Erlanger 1930; 1935) The soft genera comprised the enharmonic and chromatic, those in which the largest interval is greater than the sum of the two smaller ones, or equivalently, is greater than one half of the perfect fourth. The firm genera consisted of the diatonic, including a subclass of reduplicated forms containing repeated whole tone intervals. These main genera were further subdivided according to whether the pykna were linearly divided into approximately equal (1:1) or unequal (1:2) parts. The 1:1 divisions were termed "weak" and the 1:2 divisions, "strong."

These theorists added many new tunings to the corpus of known tetrachords and also tabulated the intervallic permutations of the genera. This led to compendious tables which may or may not have reflected actual musical practice.

Crocker's tetrachordal comparisons

Richard L. Crocker (1963, 1964, 1966) analyzed the most important of the ancient Greek tetrachords (see chapters 2 and 3) in terms of the relative magnitudes of their intervals. Crocker was interested in the relation of the older Pythagorean tuning to the innovations of Archytas and Aristoxenos. He stressed the particular emphasis placed on the position of the lichanos by Archytas who employed 28/27 as the first interval (parhypate to 1/1) in all three genera. In Pythagorean tuning, the chromatic and diatonic parhypatai are a limma (256/243, 90 cents) above hypate, while the enharmonic division is not certain. The evidence suggests a ***limmatic pyknon***, but it may not have been consistently divided much prior to the time of Archytas (Winnington-Ingram 1928).

Archytas's divisions are in marked contrast to the genera of Aristoxenos, who allowed both lichanos and parhypate to vary within considerable ranges. With Archytas the parhypatai are fixed and all the distinction between the genera is carried by the lichanoi. These relations can be seen most clearly in 5-1, 5-2, and 5-3. These figures have been redrawn from those in Crocker (1966).

This type of comparison has been extended to the genera of Didymos, Eratosthenes and Ptolemy in 5-4, 5-5, and 5-6. The genera of Didymos and Eratosthenes resemble those of Aristoxenos with their pykna divided in rough equality.

Ptolemy's divisions are quite different. For Aristoxenos, Didymos, and Eratosthenes, the ratio of the intervals of the pyknon are roughly 1:1, except in the diatonic genera. Ptolemy, however, uses approximately a 2:1 relationship.

Barbera's rate of change function

C. André Barbera (1978) examined these relations in more detail. He was especially interested in the relations between the change in the position of the lichanoi compared to the change in the position of the parhypatai as one moved from the enharmonic through the chromatic to the diatonic genera. Accordingly, he defined a function over pairs of genera which compared the change in the location of the lichanoi to the change in that of the parhypatai. His function is $(\text{lichanos}_2 - \text{lichanos}_1) / (\text{parhypate}_2 - \text{parhypate}_1)$ where the corresponding notes of two tetrachords are subscripted. This function is meaningful only when computed on a series of related genera

5-1. *Archytas's genera. These genera have a constant 28/27 as their parhypate.*

ENHARMONIC

28/27 36/35 5/4

0 63 112 498

CHROMATIC

28/27 243/224 32/27

0 63 204 498

DIATONIC

28/27 8/7 9/8

0 63 294 498

5-2. *Pythagorean genera. These genera are traditionally attributed to Pythagoras, but in fact are of Babylonian origin (Duchesne-Guillemin 1963, 1969). The division of the enharmonic pyknon is not known, but several plausible tunings are listed in the Main Catalog.*

ENHARMONIC

? ? 81/64

0 ? 90 498

CHROMATIC

256/243 2187/2048 32/27

0 90 204 498

DIATONIC

256/243 9/8 9/8

0 90 294 498

5-3. *Aristoxenos's genera, expressed in Cleonides's parts rather than ratios. One part equals 16.667 cents.*

ENHARMONIC

0 50 100 500

3 + 3 + 24 PARTS

SOFT CHROMATIC

0 67 133 500

4 + 4 + 22 PARTS

HEMIOLIC CHROMATIC

0 75 150 500

4.5 + 4.5 + 21 PARTS

INTENSE CHROMATIC

0 100 200 500

6 + 6 + 18 PARTS

SOFT DIATONIC

0 100 250 500

6 + 9 + 15 PARTS

INTENSE DIATONIC

0 100 300 500

6 + 12 + 12 PARTS

5-4. Didymos's genera. Didymos's chromatic is probably the most consonant tuning for the 6/5 genus. His diatonic differs from Ptolemy's only in the order of the 9/8 and 10/9.

ENHARMONIC

32/31	31/30	5/4	
0	55	112	498

CHROMATIC

16/15	25/24	6/5	
0	112	183	498

DIATONIC

16/15	10/9	9/8	
0	112	294	498

5-5. Eratosthenes's genera. Eratosthenes's diatonic is the same as Ptolemy's ditone diatonic.

ENHARMONIC

40/39	39/38	19/15	
0	44	89	498

CHROMATIC

20/19	19/18	6/5	
0	89	183	498

DIATONIC

256/243	9/8	9/8	
0	90	294	498

5-6. Ptolemy's genera. Only Ptolemy's own genera are shown. Ptolemy's tonic diatonic is the same as Archytas's diatonic. His ditone diatonic is the Pythagorean diatonic.

ENHARMONIC

46/45	24/23	5/4	
0	38	113	498

SOFT CHROMATIC

28/27	15/14	6/5	
0	63	182	498

INTENSE CHROMATIC

22/21	12/11	7/6	
0	81	232	498

SOFT DIATONIC

21/20	10/9	8/7	
0	85	267	498

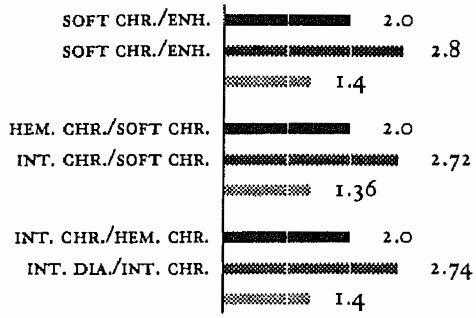
INTENSE DIATONIC

16/15	9/8	10/9	
0	112	316	498

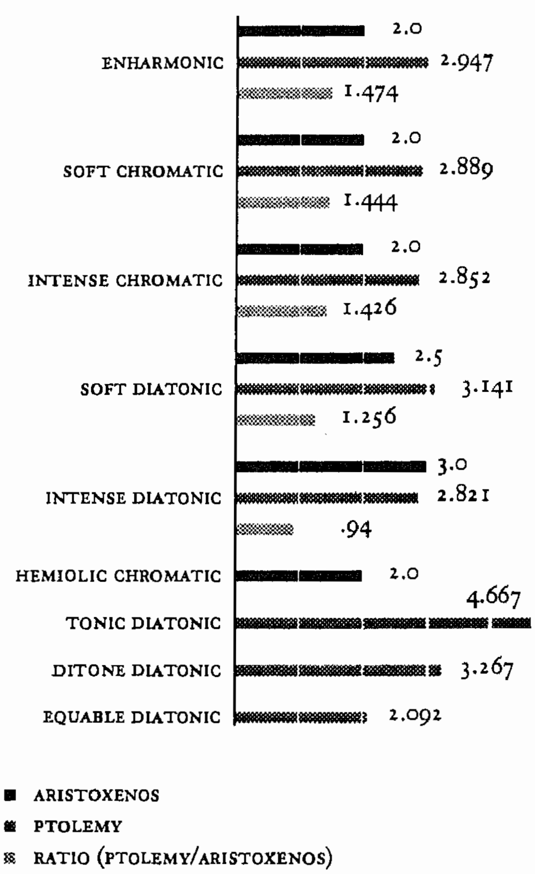
EQUABLE DIATONIC

12/11	11/10	10/9	
0	151	316	498

5-7. Barbera's function applied to Aristoxenos's and Ptolemy's genera.



5-8. Ratio of lichanos to parhypate in Aristoxenos's and Ptolemy's genera.



such as Aristoxenos's enharmonic and his chromatics or on the corresponding ones of Ptolemy. The extent to which such calculations give consistent values is a measure of the relatedness of the tetrachordal sets.

In 5-7, the results of such calculations are shown. The value for Aristoxenos's non-diatonic genera is 2.0. Ptolemy's genera yield values near 3.0, and the discrepancies are due to his use of superparticular ratios and just intonation rather than equal temperament. The proportion of the Ptolemaic to the Aristoxenian values is near 1.4.

These facts suggest that both theorists conceived their tetrachords as internally related sets, not as isolated tunings. Presumably, the increase from 2.0 to about 3 of this parameter reflects a change in musical taste in the nearly 500 years elapsed between Aristoxenos and Ptolemy.

Both ancient theorists presented additional genera not used in this computation. Some, such as Aristoxenos's hemiolic chromatic or Ptolemy's equable diatonic, had no counterpart in the other set. Ptolemy's soft diatonic appears to be only a variation or inflection of his intense (syntonic) chromatic. His remaining two diatonics, the tonic and ditonic, were of historical origin and not of his invention. The same is true of Aristoxenos's intense diatonic which seems clearly intended to represent the archaic ditone or Pythagorean diatonic.

A comparison of the corresponding members of these two authors' sets of tetrachords by a simpler function is also illuminating. If one plots the ratio of lichanos to parhypate or, equivalently, the first interval versus the sum of the first two, it is evident that Aristoxenos preferred an equal division of the pyknon and Ptolemy an unequal 1:2 relation. These preferences are shown by the data in 5-8, where the lichanos/parhypate ratio is 2.0 for Aristoxenos's tetrachords and about 3.0 for Ptolemy's non-diatonic genera.

One may wonder whether Ptolemy's tetrachords are theoretical innovations or whether they faithfully reflect the music practice of second century Alexandria. The divisions of Didymos and Eratosthenes, authors who lived between the time of Aristoxenos and Ptolemy, resemble Aristoxenos's, and there are strong reasons to assume that Aristoxenos is a trustworthy authority on the music of his period (chapter 3). The lyra and kithara scales he reports as being in use by contemporary musicians would seem to indicate that the unequally divided pyknon was a musical reality (chapter 6). Ptolemy's enharmonic does seem to be a speculative construct as the enharmonic genus was extinct by the third century BCE (Winnington-Ingram 1932). His equable diatonic, however, resembles modern Islamic scales and certain Greek orthodox liturgical tetrachords (chapter 3).

These historical studies are important not only for what they reveal about ancient musical thought but also because they are precedents for organizing groups of tetrachords into structurally related sets. The use of constant or contrasting pyknotic/apyknotic proportions can be musically significant. Modulation of genus (μεταβολε κατα γενοσ) from diatonic to chromatic or enharmonic and back was a significant stylistic feature of ancient music according to the theorists. Several illustrations of this technique are found among the surviving fragments of Greek music (Winnington-Ingram 1936).

Neo-Aristoxenian classification

The large number of new tetrachordal divisions generated by the methods of chapter 4 indicates a need for new classification tools. A conveniently simple scheme is the neo-Aristoxenian classification which assumes a tempered fourth of 500 cents and categorizes tetrachords into four classes according to the sizes of their CIs. For tetrachords in just intonation, the fourth has 498.045 cents, and the boundaries between categories will be slightly adjusted. The essential feature of this scheme is the geometrical approach of chapter three.

Those new genera whose CIs fall between a major third and perfect fourth may be denoted *hyperenharmonic* after Ervin Wilson (personal communication) who first applied it to the 56/55 · 55/54 · 9/7 genus. The hyperenharmonic CIs range from roughly 450 cents down to 425 cents. The next class is the enharmonic with CIs ranging from 425 to 375 cents, a span of 50 cents. The widest division is the chromatic, from 375 cents to 250 cents as it includes CIs whose widths vary from the neutral thirds of approximately 360–350 cents (16/13, 11/9, 27/22) through the minor and subminor thirds (6/5, 7/6) to the "half-augmented seconds" (15/13, 52/45) near 250 cents. Beyond this limit, a pyknon no longer exists and the genera are diatonic.

This neo-Aristoxenian classification is summarized in 5-9. The limits of the categories are illustrated with representative tetrachords in just intonation.

5-9. *Neo-Aristoxenian classification.* a + b + c = 500 *cents. This classification is based on the size of the largest or characteristic interval (CI); the equal division of the pyknon* (a+b) *is only illustrative and other divisions exist. The hyperenharmonic genera have CIs between the major third and the fourth and pyknotic intervals of commatic size. The enharmonic genera contain CIs approximating major thirds. The chromatic genera range from the soft chromatic to the soft diatonic of Aristoxenos or the intense chromatic of of Ptolemy. The diatonic are all those genera without pykna, i.e., whose largest interval is less than 250 cents.*

HYPERENHARMONIC

$c/10 < a + b \le 3c/17$

23+23+454 *to* 37.5+37.5+425 *cents*

80/79 · 79/78 · 13/10 *to* 50/49 · 49/48 · 32/25

ENHARMONIC

$3c/17 < a + b \le c/3$

37.5+37.5+425 *to* 62.5+62.5+375 *cents*

48/47 · 47/46 · 23/18 *to* 30/29 · 29/28 · 56/45

CHROMATIC

$c/3 < a + b \le c$

62.5+62.5+375 *to* 125+125+250 *cents*

29/28 · 28/27 · 36/29 *to* 15/14 · 14/13 · 52/45

DIATONIC

$c < a + b \le 2c$

125+125+250 *to* 167+167+167 *cents*

104/97 · 97/90 · 15/13 *to* 11/10 · 11/10 · 400/363

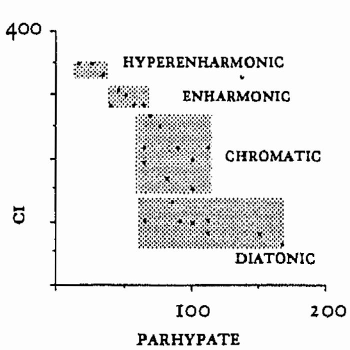
These four main classes may be further subdivided according to the proportions of the two intervals which divide the pyknon, or apyknon in the case of the diatonic genera. Because of the large number of possible divisions, it is clearer and easier to display the various subgenera graphically than to try to name them individually. Thus a number of representative tetrachords from the Main Catalog have been plotted in 5-10–12 to illustrate the most important types.

In 5-10, the first interval, as defined by the position of the note *parhypate*, has been plotted against the characteristic interval. For most of the historical tetrachords of chapters 2 and 3, this is equivalent to plotting the smallest versus the largest intervals or the first against the third. The exceptions, of course, are Archytas's enharmonic and diatonic and Didymos's chromatic.

5-11 shows the position of the third note, lichanos, graphed against the second, parhypate. This is equivalent to comparing the size of the whole pyknon (or *apyknon*) to its first interval. This particular display recalls the Greek classification by the position of the lichanoi and the differentiation into shades or chroai by the position of the parhypatai.

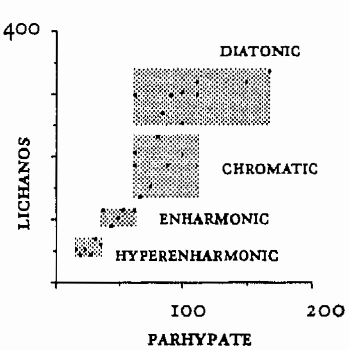
The first interval is plotted against the second in 5-12. In this graph, however, all of the permutations of this set of typical tetrachords are also plotted. This type of plot reveals the inequality of intervallic size between genera and distinguishes between permutations when the tetrachords are not in the standard Greek ascending order of smallest, medium, and large.

5-10.



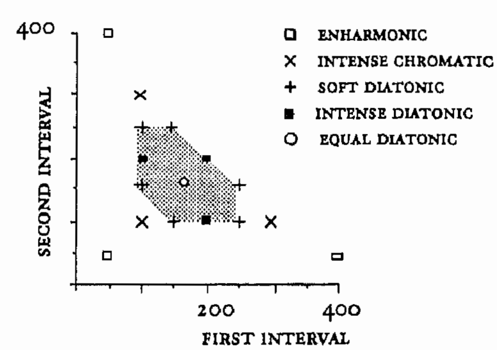
5-10. *Plot of characteristic intervals versus parhypatai. The four notes of the illustrative meson tetrachord in ascending order of pitch are hypate, parhypate, lichanos, and mese. The CI is the interval between lichanos and mese.*

5-11.



5-11. *Plot of lichanoi versus parhypatai.*

5-12.



5-12. *First interval plotted against second intervals of major tetrachordal genera. The tetrachords plotted here are 50 + 50 + 400, 100 + 100 + 300, 100 + 150 + 250, 100 + 200 + 200, and 166.67 + 166.67 + 166.67 cents in all of their intervallic permutations. The permutations of the soft diatonic genus delineate the region of Rothenberg-proper diatonic scales.*

5-13. *Intervallic inequality functions on just and tempered tetrachords.*

RATIOS	CI/MIN	CI/MID	MID/MIN
HYPERENHARMONIC			
56/55 · 55/54 · 9/7	13.95	13.70	1.018
ENHARMONIC			
28/27 · 36/35 · 5/4	7.921	66.136	1.291
32/31 · 31/30 · 5/4	7.028	6.805	1.033
46/45 · 24/23 · 5/4	10.15	5.243	1.936
CHROMATIC			
20/19 · 19/18 · 6/5	3.554	3.372	1.054
28/27 · 15/14 · 6/5	5.013	2.642	1.897
26/25 · 25/24 · 16/13	5.294	5.086	1.041
39/38 · 19/18 · 16/13	7.994	3.840	2.081
24/23 · 23/22 · 11/9	4.715	4.514	1.044
34/33 · 18/17 · 11/9	6.722	3.511	1.915
16/15 · 15/14 · 7/6	2.389	2.234	1.069
22/21 · 12/11 · 7/6	3.314	1.772	1.870
DIATONIC			
14/13 · 13/12 · 8/7	1.802	1.668	1.080
21/20 · 10/9 · 8/7	2.737	1.267	2.159
28/27 · 9/8 · 8/7	3.672	1.133	3.239
16/15 · 10/9 · 9/8	1.825	1.118	1.633
256/243 · 9/8 · 9/8	2.260	1.000	2.260
12/11 · 11/10 · 10/9	1.211	1.105	1.095
TEMPERED TETRACHORDS			
50 + 50 + 400	8.00	8.00	1.00
66.67 + 133.33 + 300	4.50	2.25	2.00
100 + 100 + 300	3.00	3.00	1.00
100 + 150 + 250	2.50	1.67	1.50
100 + 200 + 200	2.00	1.00	2.00
166.67 + 166.67 + 166.67	1.00	1.00	1.00

Intervallic inequality functions

More quantitative measures of intervallic inequality are seen in 5-13. The first measure is the ratio of the logarithms of the largest interval to that of the smallest. In practice, cents or logarithms to any base may be used. This ratio measures the extremes of intervallic inequality. The second measure is the ratio of the largest to the middle-sized interval. For tetrachords with reduplicated intervals, i.e., 256/243 · 9/8 · 9/8 or 16/15 · 16/15 · 75/64, the middle-sized interval is the reduplicated one, and this function is equal to one of the other two functions. The third measure is the ratio of the middle-sized interval to the smallest. This function often indicates the relative sizes of the two intervals of the pyknon and distinguishes subgenera with the same CI.

These functions measure the degree of inequality of the three intervals and may be defined for tetrachords in equal temperament as well as in just intonation. All of these functions are invariant under permutation of intervallic order.

Harmonic complexity functions

In addition to being classified by intervallic size, tetrachords may also be characterized by their harmonic properties. Although harmony in the sense of chords and chordal sequences is discussed in detail in chapter 7, it is appropriate in this chapter to discuss the harmonic properties of the tetrachordal intervals in terms of the prime numbers which define them.

The simplest harmonic function which may be defined on a tetrachord or over a set of tetrachords is the largest prime function. The value of this function is that of the largest prime number greater than 2 in the numerators or denominators of three ratios defining the tetrachord. The tetrachord (or any other set of intervals) is said to have an *n-limit* or be an *n*-limit construct when *n* is the largest prime number in the defining ratio(s), irrespective of its exponent and the exponent's sign.

One limitation of the *n*-limit function is that it uses only a small part of the information in the tetrachordal intervals. As a result, numerous genera with different melodic properties have the same *n*-limit. However, this one-dimensional descriptor is often used by composers of music in just intonation (David Doty, personal communication). For example, the following diverse set of tetrachords all contain 5 as their largest prime number: 25/24 · 128/125 · 5/4, 256/243 · 81/80 · 5/4, 16/15 · 25/24 · 6/5, 256/243 ·

5-14. *Harmonic complexity and simplicity functions on tetrachords in just intonation. (1) CI complexity: the sum of the prime factors of the largest interval. (2) Pyknotic complexity: the joint complexity of the two intervals of the pyknon. (3) Average complexity: the arithmetic mean of the CI and pyknotic complexities. (4) Total complexity: the joint complexity of the entire tetrachord. (5) Harmonic simplicity: 1 over the sum of the prime factors greater than 2 of the ratio defining the CI. It has been normalized by dividing by 0.2, as the maximum value of the unscaled function is 0.2, corresponding to 5/4 whose Wilson's complexity is 5.*

RATIOS	1	2	3	4	5
HYPERENHARMONIC					
56/55 · 55/54 · 9/7	13	32	22.5	32	.3846
ENHARMONIC					
28/27 · 36/35 · 5/4	5	21	13	21	1.000
32/31 · 31/30 · 5/4	5	39	22	39	1.000
46/45 · 24/23 · 5/4	5	34	19.5	34	1.000
CHROMATIC					
20/19 · 19/18 · 6/5	8	30	19	30	.6250
28/27 · 15/14 · 6/5	8	21	14.5	21	.6250
26/25 · 25/24 · 16/13	13	26	19.5	26	.3846
39/38 · 19/18 · 16/13	13	38	25.5	38	.3846
24/23 · 23/22 · 11/9	17	37	27	40	.2941
34/33 · 18/17 · 11/9	17	34	25.5	34	.2941
16/15 · 15/14 · 7/6	10	15	12.5	15	.5000
22/21 · 12/11 · 7/6	10	21	15.5	21	.5000
DIATONIC					
14/13 · 13/12 · 8/7	7	23	15	23	.7143
21/20 · 10/9 · 8/7	7	18	12.5	18	.7143
28/27 · 9/8 · 8/7	7	16	11.5	16	.7143
16/15 · 10/9 · 9/8	6	11	8.5	11	.8333
256/243 · 9/8 · 9/8	6	15	10.5	15	.8333
12/11 · 11/10 · 10/9	11	19	15	22	.4545

135/128 · 6/5, 16/15 · 75/64 · 16/15, 10/9 · 10/9 · 27/25, and 16/15 · 9/8 · 10/9. Similarly, all the Pythagorean tunings in the Catalog are at the 3-limit.

The second limitation of the largest prime number function when applied to the whole tetrachord is that it does does not distinguish between intervals which may be of differing harmonic importance to the composer. Primary distinctions between genera are determined by the sizes of their characteristic intervals. Genera with similarly sized CIs may have quite different musical effects due to the different degrees of consonance of these intervals. Similar effects are seen with the pyknotic intervals as well, particularly those due to the first interval which combines with mese or the added note, hyperhypate, to form an interval characteristic of the oldest Greek styles (Winnington-Ingram 1936 and chapter 6). In these cases, the largest prime function must be applied to the individual intervals and not just to the tetrachord as a whole.

For these reasons, other indices of harmonic complexity have been developed which utilize more of the information latent in the tetrachordal intervals. These indices have been computed on a representative set of tetrachords and their component intervals. The first of the indices is Wilson's *complexity* function which for single intervals may be defined as the sum of their prime factors (greater than 2) times the absolute values of their exponents. For example, the complexities of 3/2 and 4/3 are both 3 and those of 6/5 and 5/3 are both 8 (3 + 5). Similarly, the intervals 9/7 and 14/9 both have complexities of 13 (3 + 3 + 7). The complexities of the CIs of some important genera are tabulated in 5-14.

Wilson's complexity function may also be applied to sets of intervals by finding the modified least common multiple of the prime factors (with all the exponents made positive). The pyknon of Archytas's enharmonic consists of the intervals 28/27 and 36/35. The first ratio may be expressed as $7 + 3^3$ and the second as $3^2 + 5 + 7$. The modified least common multiple of this set is $3^3 \cdot 5 \cdot 7$ and the Wilson's complexity is 21 (3 + 3 + 3 + 5 + 7). The average complexity, which is the arithmetic mean of the complexities of the CI and the pyknon, and the total complexity, which is the joint complexity of all three intervals, are also shown in 5-14. In most cases the latter index equals the pyknotic complexity.

An alternative index which may be more convenient in some cases is the harmonic *simplicity*, which is the reciprocal of the complexity. This function may be normalized, as it is in 5-14, by dividing its values by 5, which is the maximum simplicity of a CI or tetrachord (because 5/4 is the simplest interval smaller than 4/3).

Euclidean distances between tetrachords

The methods described in chapter 4 and in the compilations of the historical authors provide many tetrachords with diverse melodic characteristics. To bring some order to these resources, some measure of the perceptual distance between different genera or between different permutations of the same genus is desirable. While a useful measure of the distance between genera may be obtained from the differences between the characteristic intervals, this measure does not distinguish between the subgenera (i.e., the 1:1 and 1:2 divisions of the pyknon). A more precise measure is afforded by the Euclidean distances between genera on a plot of the CI versus the

5-15. Euclidean distances between genera in just intonation. The upper set of numbers is the distance calculated on the largest versus the smallest intervals of the tetrachords. The lower set is computed from the first and second intervals. The Euclidean distance is the square root of the sum of the squares of the differences between corresponding intervals. Values are in cents.

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	72.09 70.67	73.99 63.43	123.59 103.37	192.96 162.62	227.94 145.59
28/27 · 15/14 · 6/5		7.71 10.91	51.84 35.81	121.91 97.54	159.50 98.81
25/24 · 16/15 · 6/5			49.76 40.14	119.04 100.91	155.39 96.09
22/21 · 12/11 · 7/6				70.26 61.73	109.77 71.56
16/15 · 9/8 · 10/9					44.45 55.02

5-16. Euclidean distances between tempered genera. The 1:2 chromatic is the "strong" form corresponding to the intense chromatic of Aristoxenos. The equal diatonic is 166.67 + 166.67 + 166.67 cents.

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	101.36 84.89	111.80 70.71	158.11 111.80	206.16 158.11	260.87 164.99
1:2 CHROMATIC (67 + 133 + 300)		33.33 47.14	60.09 37.27	105.41 74.54	166.67 105.41
INTENSE CHROMATIC (100 + 100 + 300)			50.0 50.0	100.0 100.0	149.07 94.28
SOFT DIATONIC (100 + 150 + 250)				50.0 50.0	106.72 68.72
INTENSE DIATONIC (100 + 200 + 200)					74.54 74.54

5-17. *Euclidean distances between permutations of Archytas's enharmonic genus. The function tabulated is the distance calculated on the plot of the first by the second interval of the tetrachord. The other distance function, computed from the graph of the greatest versus the least interval, is always zero between permutations of the same genus.*

5-18. *Euclidean distances between permutations of tempered genera.*

smallest interval or of the first versus the second interval.

The distances are calculated according to the Pythagorean relation: the distance is defined as the square root of the sum of the squares of the differences of the coordinates. The Euclidean distance is $\sqrt{[(CI_2 - CI_1)^2 + (\text{parhypate}_2 - \text{paryhypate}_1)^2]}$ in the first case and $\sqrt{[(\text{first interval}_2 - \text{first interval}_1)^2 + (\text{second interval}_2 - \text{second interval}_1)^2]}$ in the second. It is convenient to convert the ratios into cents for these calculations. The distances between some representative tetrachords in just intonation are tabulated in 5-15 and some in equal temperament with similar melodic contours in 5-16.

One may also use the second Euclidean distance function to distinguish between permutations of tetrachords as shown in 5-17 and 5-18.

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	337.54	337.84	20.07	323.66	323.35
28/27 · 5/4 · 36/35		14.19	323.66	457.29	467.43
36/35 · 5/4 · 28/27			323.55	467.43	155.39
36/35 · 28/27 · 5/4				337.54	337.84
5/4 · 28/27 · 36/35					14.19

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	350.0	350.0
50 + 400 + 50		494.97

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	200.0	200.0
100 + 300 + 100		282.84

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	141.42	100.0
200 + 100 + 200		100.0

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	100.0	70.71	111.81	158.11	150.0
100 + 250 + 150		158.11	50.0	212.13	180.28
150 + 100 + 250			150.0	100.0	111.80
150 + 250 + 100				180.28	141.42
250 + 100 + 150					50.0

Minkowskian distances between tetrachords

The closely related *Minkowski metric* or *city block* distance function is shown in 5-19 and 5-20 for the same sets of tetrachords. The two functions shown here are defined as the sum of the absolute values of the differences between corresponding intervals. For the upper set of numbers, the function is $(| CI_2 - CI_1 | + | parhypate_2 - paryhypate_1 |)$ and for the lower set, $(| \text{first interval}_2 - \text{first interval}_1 | + | \text{second interval}_2 - \text{second interval}_1 |)$. These computations have also been done in cents throughout for ease of comparison.

The distances between permutations may also be compared by means of the second distance function (5-21 and 5-22).

5-19. *Minkowski or "city block" distances between genera in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	84.86 70.67	92.57 70.67	151.21 119.44	245.36 203.91	305.78 203.91
28/27 · 15/14 · 6/5		7.71 15.42	66.35 48.77	160.50 133.24	220.91 133.24
25/24 · 16/15 · 6/5			58.64 48.77	152.79 133.24	213.20 133.24
22/21 · 12/11 · 7/6				94.16 84.47	109.77 84.47
16/15 · 9/8 · 10/9					77.81 60.41

5-20. *Minkowski or "city block" distances between tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	116.67 100.0	150.0 100.0	200.0 150.0	250.0 200.0	350.0 233.33
1:2 CHROMATIC (67 + 133 + 300)		33.33 66.67	83.33 50.0	133.33 100.0	233.33 200.0
INTENSE CHROMATIC (100 + 100 + 300)			50.0 50.0	100.0 100.0	200.0 133.33
SOFT DIATONIC (100 + 150 + 250)				50.0 50.0	150. 83.33
INTENSE DIATONIC (100 + 200 + 200)					100.0 100.0

5-21. *Minkowski or "city block" distances between permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	337.54	351.73	28.38	337.54	323.35
28/27 · 15/14 · 6/5		14.19	337.54	646.71	660.90
25/24 · 16/15 · 6/5			323.35	660.90	675.09
22/21 · 12/11 · 7/6				337.54	351.73
16/15 · 9/8 · 10/9					14.19

5-22. *Minkowski or "city block" distances between permutations of tempered genera.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	350.0	350.0
100 + 250 + 150		700.0

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	200.0	200.0
100 + 300 + 100		400.0

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	200.0	100.0
200 + 100 + 200		100.0

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	100.0	100.0	150.0	200.0	150.0
100 + 250 + 150		200.0	50.0	300.0	250.0
150 + 100 + 250			150.0	100.0	150.0
150 + 250 + 100				250.0	200.0
250 + 100 + 150					50.0

5-23. *Tenney pitch and harmonic distance funcions on the intervals of tetrachords in just intonation.*

	C.I.'S	MID	SMALL
56/55 · 55/54 · 9/7	0.109 1.799	.0080 3.473	.0078 3.489
28/27 · 36/35 · 5/4	.0969 1.301	.0158 2.879	.0122 3.100
32/31 · 31/30 · 5/4	.0969 1.301	.0142 2.968	.0138 2.997
46/45 · 24/23 · 5/4	.0969 1.301	.0184 2.742	.0096 3.156
20/19 · 19/18 · 6/5	.0792 1.477	.0235 2.534	.0223 2.580
28/27 · 15/14 · 6/5	.0792 1.477	.0300 2.322	.0158 2.878
26/25 · 25/24 · 16/13	.0902 2.318	.0177 2.778	.0170 2.813
39/38 · 19/18 · 16/13	.0902 2.318	.0235 2.534	.0113 3.171
24/23 · 23/22 · 11/9	.0872 1.996	.0193 2.704	.0185 2.742
34/33 · 18/17 · 11/9	.0872 1.996	.0248 2.486	.0130 3.050
16/15 · 15/14 · 7/6	.0669 1.623	.0300 2.322	.0280 2.380
22/21 · 12/11 · 7/6	.0669 1.623	.0378 2.121	.0202 2.664
14/13 · 13/12 · 8/7	.0580 1.748	.0348 2.193	.0322 2.260
21/20 · 10/9 · 8/7	.0580 1.748	.0458 1.954	.0212 2.623
28/27 · 9/8 · 8/7	.0580 1.748	.0512 1.857	1.580 2.879
16/15 · 10/9 · 9/8	.0511 1.857	.0458 1.954	.0280 2.380
256/243 · 9/8 · 9/8	.0511 1.857	.0511 1.857	.0226 4.794
12/11 · 11/10 · 10/9	.0458 1.954	.0414 2.041	.0378 2.121

Tenney's pitch and harmonic distance functions

The composer James Tenney has developed two functions to compare intervals (Tenney 1984), and has used these functions in composition, particularly in *Changes: Sixty-four Studies for Six Harps*. The first function is the *pitch-distance* function defined as the base-2 logarithm of *a/b* where *a* and *b* are the numerator and denominator respectively of the interval in an extended just intonation. This function is equivalent to Ellis's cents which are 1200 times the base-2 logarithm. The second function is his *harmonic distance*, defined as the logarithm of $a \cdot b$. This distance function is a special use of the Minkowski metric in a tonal space where the units along each of the axes are the logarithms of prime numbers. Thus the pitch distance of the interval 9/7 is log (9/7) and the harmonic distance is 2 · log (3) + log (7).

These functions may be used to characterize tetrachords by computing distances for each of the three intervals. This has been done for the set of representative tetrachords in 5-23. The upper set of numbers is the pitch distances; the lower, the harmonic distances. Alternatively, one could also apply it to the notes of the tetrachord after fixing the tonic and calculating the notes from the successive intervals.

By a slight extension of the definition, the pitch distance function may also be applied to tempered intervals. The pitch distance is the tempered interval expressed as a logarithm. For intervals expressed in cents, the formula is pitch distance = cents / 1200 log (2); other logarithmic measures could be used. This function will be most interesting for intervals which are close approximations to those in just intonation. The harmonic distance function is not well defined for tempered intervals unless they closely approximate just intervals.

The Tenney functions also may be used to measure the distance between tetrachords. The pitch distance between the CIs of two genera is the logarithm of the quotient of their ratios; i.e., the pitch distance between 5/4, the CI of the enharmonic, and 6/5, the CI of the intense chromatic, is the logarithm of 25/24. The harmonic distance is the logarithm of 3/2, the product of 5/4 and 6/5.

The pitch distance and harmonic distance functions on the CIs distinguish genera quite well, though obviously not permutations of the genera. The Tenney distance functions between representative set of tetrachords in just intonation are shown in 5-24. One could also apply the

Tenney distance functions on the pyknotic intervals to distinguish subgenera with the same CI.

The distances between tetrachords in equal temperament may also be measured by the Tenney functions. The pitch distance of the CIs is simply the difference in cents or tempered degrees. The harmonic distance is the sum of the CIs. Data on representative tempered tetrachords are shown in 5-25.

5-24. *Tenney pitch and harmonic distances between genera in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	.0177 .1761	.0177 .1761	.0270 .1638	.0458 .1481	.0512 .1427
28/27 · 15/14 · 6/5		0.0 .1584	.0122 .1461	.0280 .1303	.0334 .1249
25/24 · 16/15 · 6/5			.0122 .1461	.0280 .1303	.0334 .1249
22/21 · 12/11 · 7/6				.0158 .1181	.0212 .1121
16/15 · 9/8 · 10/9					.0054 .0969

5-25. *Tenney pitch and harmonic distances between tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC 50 + 50 + 400	100.0 700.0	100.0 700.0	150.0 650.0	200.0 600.0	233.33 566.67
1:2 CHROMATIC 67 + 133 + 300		0.0 600.0	50.0 550.0	100.0 500.0	133.33 466.67
INTENSE CHROMATIC 100 + 100 + 300			50.0 550.0	100.0 500.0	133.33 466.67
SOFT DIATONIC 100 + 150 + 250				50.0 450.0	83.33 416.67
INTENSE DIATONIC 100 + 200 + 200					33.33 366.67

factor of 2 · ξ(*hcf*), where *hcf* is the highest common factor, must be subtracted from the denominator of the formula.

Barlow's harmonicity function is applied to set of tetrachords in just intonation in 5-26. The harmonicities of the three intervals are computed separately. The harmonicity of 4/3 is the constant –0.2143. The harmonicities of the pykna are also included to complete the characterization of the tetrachords.

In the case of the general tetrachord $a \cdot b \cdot c$, where $c = 4/3ab$, there are four ratios, 1/1, a, $a \cdot b$, and 4/3. The $n \cdot (n-1)/2 = 6$ combinatorial intervals are a, ab, 4/3, b, $4/3a$, and $4/3ab$. For example, Archytas's enharmonic, 28/27 · 36/35 · 5/4, yields the tones 1/1, 28/27, 16/15, and 4/3. The combinatorial intervals are 28/27, 16/15, 4/3, 36/35, 9/7, and 5/4 the six non-redundant differences between the four tones of the tetrachord. The definition of these intervals for equally tempered tetrachords is shown as the Polansky set in 5-48. In just intonation, the sums and differences become products and quotients and the zero and 500 cents are replaced by 1/1 and 4/3 respectively.

For scales and other sets of ratios, Barlow defined a third function, termed *specific harmonicity*. The specific harmonicity of a set of ratios is the square of the number of tones divided by the sum of the absolute values of the reciprocals of the harmonicities of the combinatorial intervals (Barlow 1987). For the tetrachord, $n = 4$ and $n^2 = 16$. The specific harmonicities are presented in 5-27–29 for various sets of tetrachords.

Similarly, the specific harmonicities of scales generated from tetrachords may be computed. In the case of heptatonic scales, there are eight tones including the octave (2/1) and 28 combinatorial relations, which are defined analogously to the six of the tetrachord. The specific harmonicities of the same set of tetrachords as in 5-26 are given in 5-27. The specific harmonicities of both the tetrachords and a representative heptatonic scale are included in this table.

The Dorian mode was selected for simplicity, but other scales could have been used as well (see chapter 6 for a detailed discussion of scale construction from tetrachords). It is the scale composed of an ascending tetrachord, a 9/8 tone, and an identical tetrachord which completes the octave. Abstractly, the tones are 1/1 a ab 4/3 3/2 $3a/2$ $3ab/2$ 2/1, where $a \cdot b \cdot 4/3ab$ is the generalized tetrachord in just intonation. The set of combinatorial intervals is a, ab, 4/3, 3/2, $3a/2$, $3ab/2$, 2/1, b, $4/3a$, $3/2a$, 3/2, $3b/2$, $2/a$, $4/3ab$, $3/2ab$, $3/2b$,

5-27. *Barlow's specific harmonicity function on tetrachords and tetrachordal scales. The specific harmonicity function is the square of the number of tones in the scale divided by sum of the reciprocals of the harmonicities of the combinatorial intervals (Barlow 1987) without regard to sign. For the tetrachord, the number of tones is 4,* $n^2 = 16$*, and there are six combinatorial intervals (see 5-28). The specific harmonicity of the Dorian mode is defined as above save that* n = 8 *(including the octave),* $n^2 = 64$*, and there are 28 intervals* (n · (n-1)/2).

	RATIOS	TETRACHORD	DORIAN
1.	56/55 · 55/54 · 9/7	.1063	.0973
2.	28/27 · 36/35 · 5/4	.1859	.1633
3.	32/31 · 31/30 · 5/4	.0724	.0660
4.	46/45 · 24/23 · 5/4	.0885	.0815
5.	20/19 · 19/18 · 6/5	.1042	.0946
6.	28/27 · 15/14 · 6/5	.1911	.1721
7.	26/25 · 25/24 · 16/13	.1062	.0998
8.	39/38 · 19/18 · 16/13	.0719	.0677
9.	24/23 · 23/22 · 11/9	.0767	.0698
10.	34/33 · 18/17 · 11/9	.0848	.0807
11.	16/15 · 15/14 · 7/6	.2170	.1879
12.	22/21 · 12/11 · 7/6	.1375	.1274
13.	14/13 · 13/12 · 8/7	.1247	.1143
14.	21/20 · 10/9 · 8/7	.1739	.1627
15.	28/27 · 9/8 · 8/7	.2101	.1885
16.	16/15 · 10/9 · 9/8	.2658	.2363
17.	256/243 · 9/8 · 9/8	.2212	.2025
18.	12/11 · 11/10 · 10/9	.1609	.1437
19.	11/10 · 11/10 · 400/363	.0829	.0797
20.	16/15 · 25/24 · 6/5	.2374	.2133

3/2, 2/*ab*, 9/8, 9*a*/8, 9*ab*/8, 3/2, *a*, *ab*, 4/3, *b*, 4/3*a*, 4/3*ab*. The repeated intervals are a consequence of the modular structure of tetrachordal scales.

As can be seen from 5-27, the specific harmonicity function distinguishes different tetrachords and their derived scales quite well. 5-28 shows the results of an attempt to use this function to distinguish permutations of tetrachords from each other. Although the specific harmonicity function does not differentiate between intervallic retrogrades ($a \cdot b \cdot c$ versus $c \cdot b \cdot a$) of single tetrachords, it is quite effective when applied to the corresponding heptatonic scales.

Finally, since the specific harmonicity function is basically a theoretical measure of consonance, it would be interesting to use it to determine the most consonant tunings or shades (chroai) of the various genera. Accordingly, a number of tetrachords whose intervals had relatively "digestible" prime factors were examined. The results are tabulated in 5-29. It is clear that while the diatonic genera are generally more consonant than chromatic and they in turn are more harmonious than the enharmonic, there is considerable overlap between genera and permutations.

In particular, the most consonant chromatic genera are more consonant than many of the diatonic tunings.

5-28. *Barlow's specific harmonicity function on the permutations of Ptolemy's intense diatonic genus.*

	RATIOS	TETRACHORD	DORIAN
1.	16/15 · 9/8 · 10/9	.2794	.2567
2.	16/15 · 10/9 · 9/8	.2658	.2363
3.	9/8 · 10/9 · 16/15	.2658	.2535
4.	9/8 · 16/15 · 10/9	.2586	.2407
5.	10/9 · 16/15 · 9/8	.2586	.2398
6.	10/9 · 9/8 · 16/15	.2794	.2486

5-29. *The most consonant genera according to Barlow's specific harmonicity function.*

	RATIOS	TETRACHORD	DORIAN
	ENHARMONIC		
1A.	256/243 · 81/80 · 5/4	.1878	.1669
1B.	5/4 · 81/80 · 256/243	.1878	.1715
2A.	28/27 · 36/35 · 5/4 4	.1859	.1633
2B.	5/4 · 36/35 · 28/27	.1859	.1667
3A.	25/24 · 128/125 · 5/4	.1806	.1550
3B.	5/4 · 128/125 · 25/24	.1806	.1556
	CHROMATIC		
1A.	16/15 · 25/24 · 6/5	.2374	.2133
1B.	6/5 · 25/24 · 16/15	.2374	.2145
2.	16/15 · 75/64 · 16/15	.2317	.2008
3A.	10/9 · 81/80 · 32/27	.2290	.2046
3B.	32/27 · 81/80 · 10/9	.2290	.2035
4A.	25/24 · 27/25 · 32/27	.1926	.1745
6A.	9/8 · 64/63 · 7/6 9	.2137	.1937
6B.	7/6 · 64/63 · 9/8	.2137	.1903
7A.	10/9 · 36/35 · 7/6	.2032	.1783
7B.	7/6 · 36/35 · 10/9	.2032	.1797
	DIATONIC		
1A.	9/8 · 28/27 · 8/7	.2176	.2027
1B.	8/7 · 28/27 · 9/8	.2176	.1914
2A.	10/9 · 21/20 · 8/7	.2104	.1888
2B.	8/7 · 21/20 · 10/9	.2104	.1856
3A.	16/15 · 9/8 · 10/9	.2794	.2567
3B.	10/9 · 9/8 · 16/15	.2794	.2486
4A.	256/243 · 9/8 · 9/8	.2212	.2025
4B.	9/8 · 9/8 · 256/243	.2212	.2105
5.	10/9 · 27/25 · 10/9	.2251	.1993

Euler's *gradus suavitatis* function

A function somewhat similar to Wilson's, Tenney's, and Barlow's functions is Euler's *gradus suavitatis* (GS) or degree of harmoniousness, consonance, or pleasantness (Euler 1739 [1960]; Helmholtz [1877] 1954). Like the other functions, the GS is defined on the prime factors of ratios, scales, or chords.

Unlike Barlow's functions, the GS is very easy to compute. The GS of a prime number or of the ratio of a prime number relative to 1 is the prime number itself, i.e., the GS of 3/1 is 3. The GS of a composite number is the sum of the GSs of the prime factors minus one less than the number of factors. The GS of a ratio is found by first converting it to a section of the harmonic series and then computing the least common multiple of the terms. The GS of the least common multiple is the GS of the ratio.

Sets of ratios such as chords and scales may be converted to sections of the harmonic series by multiplying each element by the lowest common denominator. For example, the harmonic series form of the major triad

5-30. *Euler's gradus suavitatis function on tetrachords in just intonation. (1) is a hyperenharmonic genus, (2)–(4) are enharmonic, (5)–(12) and (20) are chromatic, and (13)–(19) are diatonic. The tetrachords are in their standard form with the small intervals at the base and the largest interval at the top. See 5-32 and 5-33 for other permutations of the tetrachord.*

	RATIOS	INTERVAL A	INTERVAL B	CI	PYKNON
1.	56/55 · 55/54 · 9/7	24	22	11	15 (28/27)
2.	28/27 · 36/35 · 5/4	15	17	7	11 (16/15)
3.	32/31 · 31/30 · 5/4	36	38	7	11 (16/15)
4.	46/45 · 24/23 · 5/4	32	28	7	11 (16/15)
5.	20/19 · 19/18 · 6/5	25	24	8	10 (10/9)
6.	28/27 · 15/14 · 6/5	15	14	8	10 (10/9)
7.	26/25 · 25/24 · 16/13	22	14	17	17 (13/12)
8.	39/38 · 19/18 · 16/13	34	24	17	17 (13/12)
9.	24/23 · 23/22 · 11/9	28	34	15	15 (12/11)
10.	34/33 · 18/17 · 11/9	30	22	15	15 (12/1)
11.	16/15 · 15/14 · 7/6	11	14	10	10 (8/7)
12.	22/21 · 12/11 · 7/6	20	15	10	10 (8/7)
13.	14/13 · 13/12 · 8/7	20	17	10	10 (7/6)
14.	21/20 · 10/9 · 8/7	15	10	10	10 (7/6)
15.	28/27 · 9/8 · 8/7	15	8	10	10 (7/6)
16.	16/15 · 10/9 · 9/8	11	10	8	12 (32/27)
17.	256/243 · 9/8 · 9/8	19	8	8	12 (32/27)
18.	12/11 · 11/10 · 10/9	15	16	10	8 (6/5)
19.	11/10 · 11/10 · 400/363	16	16	35	31 (121/100)
20.	16/15 · 25/24 · 6/5	11	14	8	10 (10/9)

5-31. *Euler's gradus suavitatis function on tetrachords and tetrachordal scales. (1) is a hyperenharmonic genus, (2)–(4) are enharmonic, (5)–(12) and (20) are chromatic, and (13)–(19) are diatonic. The harmonic series representation of the Dorian mode of 16/15 · 9/8 · 10/9 is 30:32:36:40:45:48:54:60. Its least common multiple is 4320 and its GS is 16.*

	RATIOS	TETRACHORD	DORIAN
1.	56/55 · 55/54 · 9/7	30	33
2.	28/27 · 36/35 · 5/4	21	24
3.	32/31 · 31/30 · 5/4	42	45
4.	46/45 · 24/23 · 5/4	35	38
5.	20/19 · 19/18 · 6/5	29	32
6.	28/27 · 15/14 · 6/5	19	22
7.	26/25 · 25/24 · 16/13	27	30
8.	39/38 · 19/18 · 16/13	39	42
9.	24/23 · 23/22 · 11/9	40	43
10.	34/33 · 18/17 · 11/9	33	36
11.	16/15 · 15/14 · 7/6	17	20
12.	22/21 · 12/11 · 7/6	22	25
13.	14/13 · 13/12 · 8/7	24	27
14.	21/20 · 10/9 · 8/7	19	23
15.	28/27 · 9/8 · 8/7	16	19
16.	16/15 · 10/9 · 9/8	16	19
17.	256/243 · 9/8 · 9/8	19	22
18.	12/11 · 11/10 · 10/9	21	24
19.	11/10 · 11/10 · 400/363	35	38
20.	16/15 · 25/24 · 6/5	17	20

1/1 5/4 3/2 is 4:5:6. The least common multiple of this series is 60 and the GS of the major scale thus is 9.

The GSs of the component intervals of the usual set of tetrachords are shown in 5-30. The GS of 1/1 is 1 and that of 4/3 is 5. In 5-31, the GSs of both the tetrachords and the Dorian mode generated from each tetrachord are tabulated. The GSs of the Dorian mode are 3 more than the GSs of the corresponding tetrachords, reflecting the structure of the mode which has the identical series of intervals repeated at the perfect fifth.

The GS seems not to be particularly useful for distinguishing permutations of tetrachords, as evidenced by 5-32. It is noteworthy that the most harmonious arrangements of Ptolemy's intense diatonic are those which generate the major and natural minor modes (see the section on tritriadic scales in chapter 7).

As with Barlow's functions, the GS ranks the enharmonic the least harmonious of the major genera, though the most consonant tunings and arrangement overlap with those of the chromatic (5-33). Similarly, the most harmonious chromatic tunings approach those of the diatonic.

Interestingly, however, the most harmonious enharmonic tuning is 28/27 · 5/4 · 36/35 and its retrograde which have the largest interval medially. The same is true for the chromatic 16/15 · 6/5 · 25/24. Of the diatonic forms, the two arrangements of Ptolemy's intense diatonic with the 9/8 medial are the most consonant.

Although the GS is an interesting and potentially useful function, it does have one weakness. Because the ratios defining small deviations from ideally consonant intervals contain either large primes or large composites, the GS of slightly mistuned consonances can become arbitrarily large. Thus the GS would predict slightly mistuned consonances to be extremely dissonant, a prediction not consistent with observation.

5-32. *Euler's gradus suavitatis function on the permutations of Ptolemy's intense diatonic genus. (1) is the prime form. (2) is the order given by Didymos.*

	RATIOS	TETRACHORD	DORIAN
1.	16/15 · 9/8 · 10/9	13	16
2.	16/15 · 10/9 · 9/8	16	19
3.	9/8 · 10/9 · 16/15	16	19
4.	9/8 · 16/15 · 10/9	16	19
5.	10/9 · 16/15 · 9/8	16	19
6.	10/9 · 9/8 · 16/15	13	16

	RATIOS	TETRACHORD	DORIAN
	ENHARMONIC		
1A.	256/243 · 81/80 · 5/4	23	26
2A.	28/27 · 36/35 · 5/4	21	24
2B.	28/27 · 5/4 · 36/35	19	22
2C.	36/35 · 28/27 · 5/4	21	24
3A.	25/24 · 128/125 · 5/4	22	25
	CHROMATIC		
1A.	16/15 · 25/24 · 6/5	17	20
1B.	25/24 · 16/15 · 6/5	18	21
1C.	16/15 · 6/5 · 25/24	16	19
2.	16/15 · 75/64 · 16/15	17	20
3A.	10/9 · 81/80 · 32/27	18	21
3B.	32/27 · 81/80 · 10/9	18	21
4A.	25/24 · 27/25 · 32/27	20	23
4B.	32/27 · 27/25 · 25/24	20	23
5A.	16/15 · 15/14 · 7/6	17	20
5B.	16/15 · 7/6 · 15/14	19	22
6A.	9/8 · 64/63 · 7/6	19	22
6B.	64/63 · 9/8 · 7/6	17	20
7A.	10/9 · 36/35 · 7/6	18	21
7B.	10/9 · 7/6 · 36/35	19	22
7C.	36/35 · 10/9 · 7/6	20	23
	DIATONIC		
1A.	9/8 · 28/27 · 8/7	18	21
1B.	8/7 · 9/8 · 28/27	16	19
2A.	10/9 · 21/20 · 8/7	18	21
2B.	21/20 · 10/9 · 8/7	19	22
3A.	16/15 · 9/8 · 10/9	13	16
3B.	10/9 · 9/8 · 16/15	13	16
4A.	256/243 · 9/8 · 9/8	19	22
5.	10/9 · 27/25 · 10/9	17	20

5-33. *The most consonant genera according to Euler's gradus suavitatis function. These ratios are the most consonant permutations of the most consonant tunings of each of the genera. In cases where the most consonant permutation according to Barlow's functions is different from the one(s) according to Euler's, both are given. The gradus suavitatis of a set of ratios is the GS of their least common multiple after the set has been transformed into a harmonic series.*

This failure, however, is a feature shared by the other simple theories of consonance based upon the prime factorization of intervals. Helmholtz's beat theory (Helmholtz [1877] 1954) and the semi-empirical "critical band" theories of Plomp and Levelt (1965) and Kameoka and Kuriyagawa (1969a, 1969b) avoid predicting infinite dissonance for mistuned consonances, but are more complex and difficult to use. The prime factor theories are adequate for theoretical work and for choosing between ideally tuned musical structures.

Statistical measures on tetrachordal space

The concepts of the degree of intervallic inequality and of the perceptual differences between tetrachords may be clarified by computing some of the standard statistical measures on a set of representative tetrachords. The arithmetic mean of the three intervals is 500/3 or 166.667 cents in equal temperament or $\sqrt[3]{(4/3)}$ in just intonation. The mean deviation, standard deviation, and variance are calculated according to the usual formulae for entire populations with $n = 3$. These data are shown in 5-34 for some representative tetrachords in just intonation and in 5-35 for a corresponding set in equal temperament. While not distinguishing permutations, these functions differentiate between genera quite well, although the degree to which the mathematical differences correlate with the perceptual is not known.

The geometric mean, harmonic mean, and root mean square (or quadratic mean) may be calculated in a similar fashion. Like the other statistical measures above, these are non-linear functions of the relative sizes of the intervals and they have considerable ability to discriminate between the various genera. The relevant data are shown in 5-36 and 5-37.

Several properties of these functions are apparent: for a given degree of intervallic asymmetry, the root mean square will show the greatest value, 5-34. *Mean deviations, standard deviations, and variances of the intervals of tetrachords in just intonation. The arithmetic mean has the constant value 166.67 cents (500/3) for all genera. In just intonation its value is the cube root of 4/3. The standard deviation and variance are computed with n=3.*

	MEAN DEV.	STANDARD DEV.	VARIANCE
28/27 · 36/35 · 5/4	146.87	155.88	24299.31
28/27 · 15/14 · 6/5	99.75	108.29	11725.73
25/24 · 16/15 · 6/5	99.75	107.12	11474.97
22/21 · 12/11 · 7/6	67.24	76.84	5904.95
16/15 · 9/8 · 10/9	36.19	39.38	1550.44
12/11 · 11/10 · 10/9	10.93	12.99	168.70

5-35. *Mean deviations, standard deviations, and variances of the intervals of tempered tetrachords.*

	MEAN DEV.	STANDARD DEV.	VARIANCE
ENHARMONIC (50 + 50 + 400)	155.56	164.99	27222.22
1:2 CHROMATIC (67 + 133 + 300)	88.89	98.13	9629.62
INTENSE CHROMATIC (100 + 100 + 300)	88.89	94.28	8888.89
SOFT DIATONIC (100 + 150 + 250)	55.56	62.36	3888.89
INTENSE DIATONIC (100 + 200 + 200)	44.44	47.14	2222.22
EQUAL DIATONIC	0.0	0.0	0.0

5-36. *Geometric mean, harmonic mean, and root mean square of the intervals of tetrachords in just intonation. For* n = 3, *the geometric mean is the cube root of* a·b·(500 − a − b)*; the harmonic mean is* $3/\Sigma(1/i)$*, where* $1/i = 1/a$, $1/b$, *and* $1/(500 - a - b)$*; the root mean square is* $\sqrt{(\Sigma(i^2)/3)}$*, where* $i^2 = a^2, b^2, (500 - a - b)^2$.

	GEOMETRIC	HARMONIC	RMS
28/27 · 36/35 · 5/4	105.86	76.97	227.73
28/27 · 15/14 · 6/5	133.40	109.40	198.21
25/24 · 16/15 · 6/5	135.58	114.21	197.58
22/21 · 12/11 · 7/6	147.90	131.57	182.94
16/15 · 9/8 · 10/9	160.77	155.15	170.62
12/11 · 11/10 · 10/9	165.51	165.01	166.52

5-37. *Geometric mean, harmonic mean, and root mean square of tempered tetrachords.*

	GEOMETRIC	HARMONIC	RMS
ENHARMONIC (50 + 50 + 400)	100.0	70.59	234.52
1:2 CHROMATIC (67 + 133 + 300)	138.79	116.38	193.41
INTENSE CHROMATIC (100 + 100 + 300)	144.23	128.57	191.41
SOFT DIATONIC (100 + 150 + 250)	155.36	145.16	177.95
INTENSE DIATONIC (100 + 200 + 200)	158.74	150.0	173.21
EQUAL DIATONIC	166.67	166.67	166.67

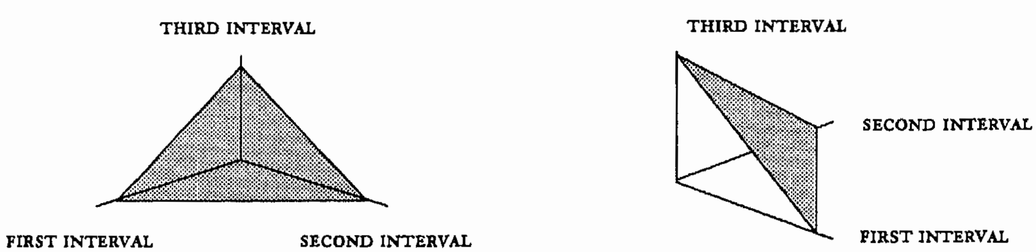
the geometric the next, and the harmonic the least, except for the arithmetic mean, which is insensitive to this parameter.

The set of all possible tetrachords instead of just representative examples or selected pairs may be studied by computing these standard statistical measures over the whole of tetrachordal space. This space may be defined by magnitudes of the first and second intervals (parhypate to hypate and lichanos to parhypate) as the third interval (mese to lichanos) is completely determined by the values of the first two.

This idea may be made clearer by plotting a simple linear function such as the third tetrachordal interval itself versus the first and second intervals. The third interval may be defined as $500 - x - y$, where x is the lowest interval and y the second lowest. The domain of this function is defined by the inequalities $0 \leq x \leq 500$ cents, $0 \leq y \leq 500$ cents, and $x + y \leq 500$ cents. 5-38 depicts the "third interval function" from two angles. Its values range from 0 to 500 cents.

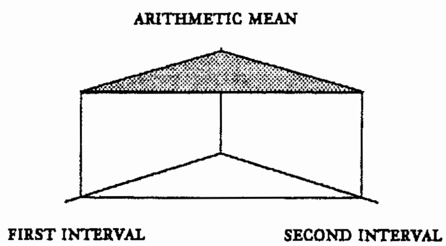
The arithmetic, geometric, harmonic, and root mean square functions are shown in 5-39 through 5-41. The arithmetic mean is a plane of constant height at 166.667 cents for all values of the three intervals. The geometric and harmonic means have dome and arch shapes respectively, while the root mean square somewhat resembles the roof of a pagoda. The shapes of these latter means may be clearer in the contour plots in the lower portions of the figures.

One may conclude that the arithmetic mean obscures the apparent distance between genera, the geometric mean reveals it, the harmonic mean maximizes it, and the root mean square exaggerates it. This conclusion is illustrated in 5-43 where a cross-section through the plot is made where the second interval has the value 166.667 cents and the first interval varies from



5-38. *The third interval function, seen frontally and obliquely. The three intervals are parhypate to hypate, lichanos to parhypate, and mese to lichanos. They always sum 500 cents (3/2 in just intonation).*

5-39. *Arithmetic mean of the three tetrachordal intervals. The arithmetic mean has the constant value of 166.67 cents. The domain of this function is the x and y axes (0 < x < 500), (0 < y < 500), and the line y = 500 − x, where x and y are the first and second intervals of the tetrachord. The third interval may also approach zero.*



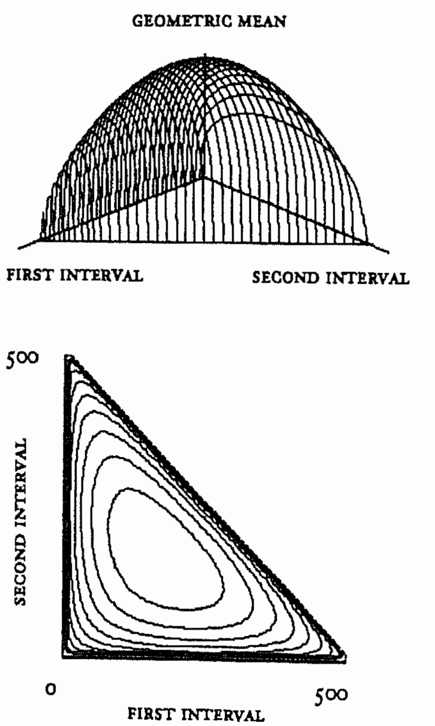
0 to 333.333 cents. The means are all equal when all three intervals of the tetrachord are 166.667 cents.

The analogous representation is applied to the mean deviation, standard deviation, and variance, which are shown in 5-44–46. The variance has been divided by 100 so that it may be plotted on the same scale as the other statistical functions.

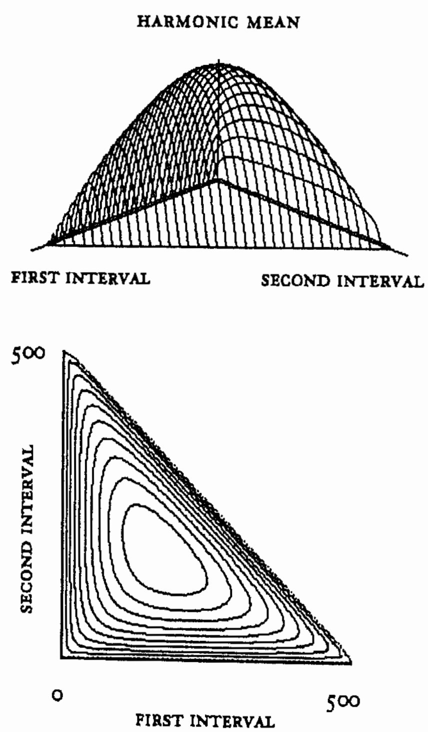
These functions have a minimum value of zero when all three intervals of the tetrachord are 166.667 cents each. This is seen most clearly in the cross-section plot of 5-47.

Based on its properties with respect to the four means and three statistical measures, the equally tempered division of the fourth appears to be a most interesting genus. It is the point where the three means are equal and where the statistical functions have their minima.

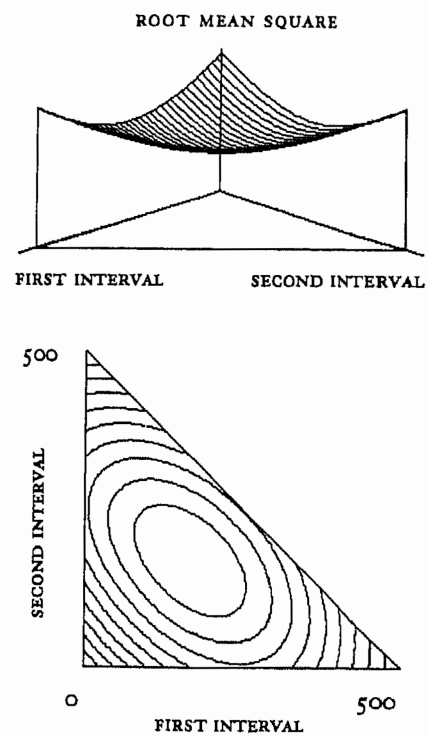
5-40. *Geometric mean of the three tetrachordal intervals.*



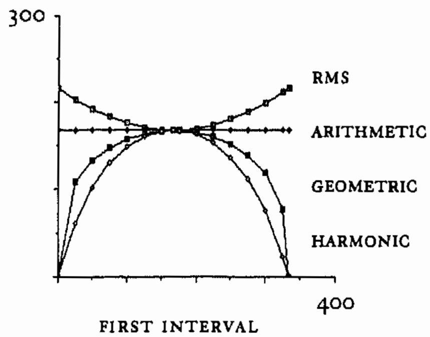
5-41. *Harmonic mean of the three tetrachordal intervals.*



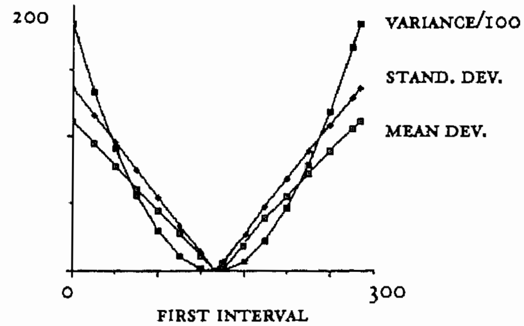
5-42. *Root mean square of the three tetrachordal intervals.*



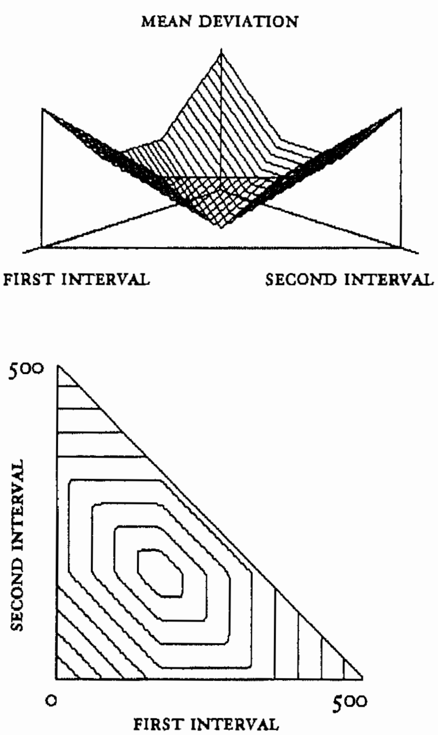
5-43. *Cross-sections of the various means of the three tetrachordal intervals when the second interval equals 166.67 cents.*



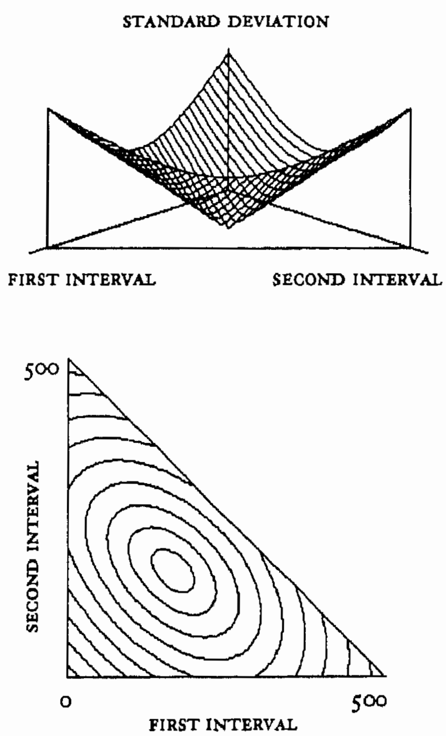
5-47. *Cross-section of the mean deviation, standard deviation, and variance of the three tetrachordal intervals when the second interval equals 166.67 cents.*



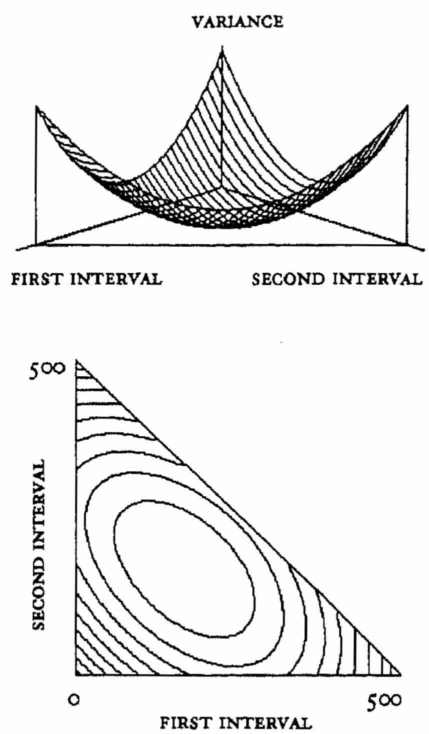
5-44. *Mean deviation of the three tetrachordal intervals.*



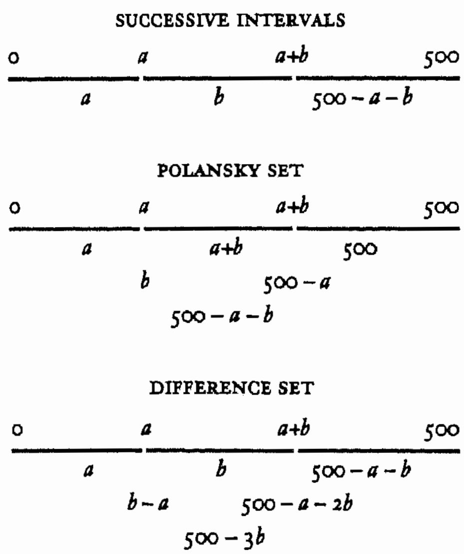
5-45. *Standard deviations of the three tetrachordal intervals.*



5-46. *Variance of the three tetrachordal intervals.*



5-48. Interval sets of the abstract tetrachord, 0 a *a+b* 500. *In just intonation the abstract tetrachord may be written 1/1* a a·b *4/3 or 0* a a+b *498 cents, and the intervals adjusted accordingly.*



Polansky's morphological metrics

A more sophisticated approach with potentially greater power to discriminate between musical structures has been taken by Larry Polansky (1987b). While designed to handle larger and more abstract sets of elements than tetrachords, i.e., the type of scale and scale-like aggregates discussed in chapters 6 and 7, and even sets of timbral, temporal, or rhythmic information, Polansky's *morphological metrics* may be applied to smaller formations as well.

Morphological metrics are distance functions computed on the notes or intervals between the notes of an ordered musical structure. A morphological metric is termed linear or combinatorial according to the number of elements or intervals used in the computations: the more intervals or elements used in the computation, the more combinatorial the metric. In other words, combinatorial metrics tend to take into account more of the relationships between component parts. A strictly linear interval set as well as two of the possible combinatorial interval sets derived from an abstract, generalized tetrachord are shown in 5-48. For a strictly linear interval set of a morphology (or scale) of length L, there are $L - 1$ intervals. The maximum combinatorial length for a morphology of length L is the binomial coefficient $(L^2 - L)/2$, notated as L_m.

The simplest of Polansky's metrics is the ordered linear absolute magnitude (OLAM) metric which is the average of the absolute value of differences between corresponding members of two tetrachords. In the case of two tetrachords spanning perfect fourths of 500 cents, this function reduces to the sum of the absolute values of the differences between the two parhypatai and the two lichanoi divided by four. Given two tetrachords $a_1 + b_1 + 500 - a_1 - b_1$ and $a_2 + b_2 + 500 - a_2 - b_2$, the equation is:

$$\sum_{i=2}^{L} | e_{1_i} - e_{2_i} | / L,$$

where $L = 4$ and $e_{n_i} = (0, a_1, a_1 + b_1, 500)$ cents and $(0, a_2, a_2 + b_2, 500)$ cents. When not divided by L, this metric is identical to the Minkowski or "city block" metric previously discussed. Note that the OLAM metric does not take intervals into account, so it looks at L rather than $L - 1$ values.

A simpler formula, $(|a_2 - a_1|$ and $|a_2 + b_2 - a_1 - b_1|) / 2$, would be defensible in this context as zero and 500 cents are constant for all tetrachords of this type. If the tetrachords are built above different tonics or their fourths spanned different magnitudes, i.e., 500 and 498 or 583, etc., the first equation must be used.

The next simplest applicable metric is the ordered linear intervallic magnitude (OLIM) metric which is the average of the absolute values of the difference between the three intervals which define the tetrachords. In the case of the two tetrachords above, the intervals are a_1, b_1, $500-a_1-b_1$ and a_2, b_2, $500-a_2-b_2$. The equation for this metric function is:

$$\left| \sum_{i=2}^{L} \left(\left| e_{1_i} - e_{1_{i-1}} \right| - \left| e_{2_i} - e_{2_{i-1}} \right| \right) \right| / (L-1),\ L-1=3,$$

5-49. Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on tetrachords in just intonation.

where i ranges from 2 through L, since intervals are being computed.

In 5-49, these two simple metrics are applied to a group of representative tetrachords in just intonation. The melodically similar tempered cases are shown in 5-50. Permutations of genera are analyzed in 5-51 and 5-52. The OLAM metric distinguishes between these genera quite well; the OLIM less so, but patterns are suggested which data on a larger set of tetrachords

5-50. Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on tempered genera.

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	17.67 47.11	19.60 47.11	34.25 79.63	63.17 135.94	72.90 135.94
28/27 · 15/14 · 6/5		1.93 5.14	16.59 32.51	45.50 88.83	55.23 88.83
25/24 · 16/15 · 6/5			14.66 32.51	43.57 88.83	53.30 88.83
22/21 · 12/11 · 7/6				28.92 56.31	38.64 56.31
16/15 · 9/8 · 10/9					9.73 25.94

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	29.17 66.67	37.50 66.67	50.0 100.0	62.50 133.33	87.50 155.56
1:2 CHROMATIC (67 + 133 + 300)		8.33 22.22	20.83 33.33	33.33 66.67	58.33 88.89
INTENSE CHROMATIC (100 + 100 + 300)			8.333 33.33	25.0 66.67	50.0 88.89
SOFT DIATONIC (100 + 150 + 250)				12.50 33.33	37.50 55.56
INTENSE DIATONIC (100 + 200 + 200)					25.0 44.44

5-51. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	84.39 225.03	84.39 225.03	3.55 9.46	165.22 225.03	161.68 215.57
28/27 · 5/4 · 36/35		7.10 9.46	87.93 225.03	80.83 215.57	84.38 225.03
36/35 · 5/4 · 28/27			80.83 215.57	87.93 225.03	84.39 225.03
36/35 · 28/27 · 5/4				225.03 225.03	165.22 225.03
5/4 · 28/27 · 36/35					3.55 9.46

5-52. *Ordered linear absolute magnitude (upper) and ordered linear intervallic magnitude (lower) metrics on permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	87.50 233.3	175.0 233.3
50 + 400 + 50		87.50 233.3

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	50.0 133.3	100.0 133.3
100 + 300 + 100		50.0 133.3

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	25.0 66.67	50.0 66.67
200 + 100 + 200		25.0 66.67

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	25.0 66.67	12.50 33.33	50.0 100.0	62.50 100.0	75.0 100.0
100 + 250 + 150		37.50 100.0	25.0 33.33	37.50 100.0	50.0 100.0
150 + 100 + 250			37.50 100.0	50.0 66.67	62.50 100.0
150 + 250 + 100				37.50 100.0	25.0 66.67
250 + 100 + 150					12.50 33.33

may reveal. In particular, the OLIM metric fails to distinguish between permutations of tempered tetrachords.

In theory, morphological metrics on combinatorial interval sets have greater discriminatory power than metrics on linear sets. Two sets of combinatorial intervals were derived from the simple successive intervals of 5-48. The first set, the Polansky set, is that described by Polansky (1987b). The second set, the difference set, was constructed from iterated differences of differences (Polansky, personal correspondence).

The ordered combinatorial intervallic magnitude (OCIM) metric is the average of the absolute value of the differences between corresponding elements of the musical structure. Its definition is:

$$\sum_{j=1}^{L-1} \sum_{i=1}^{L-j} |\Delta(e_{1_i}, e_{1_{i+j}}) - \Delta(e_{2_i}, e_{2_{i+j}})| / L_m,$$

where L_m = the number of intervals in the set (the binomial coefficient, described above). To apply it to other combinatorial interval sets, it must be appropriately modified to something like:

$$\sum_{i=2}^{L} |(I_{1_i} - I_{2_i})| / L_m,$$

where I_{n_i} are the elements of a set like the difference set of 5-48.

As can be seen in 5-53 and 5-54, the OCIM metric calculated on the two sets of intervals from these tetrachords discriminates between genera very well. Both sets of intervals are roughly equivalent with this metric.

Permutations are studied in 5-55 and 5-56. On neither interval set does the OCIM metric distinguish permutations completely.

5-53. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	35.34 94.23	36.62 86.52	62.65 141.68	110.08 223.11	116.57 184.20
28/27 · 15/14 · 6/5		3.86 10.28	27.31 47.45	74.75 128.88	81.23 104.01
25/24 · 16/15 · 6/5			26.03 55.16	15.36 136.59	79.94 106.58
22/21 · 12/11 · 7/6				47.43 81.43	53.92 61.10
16/15 · 9/8 · 10/9					19.45 51.87

5-54. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tempered tetrachords.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC	52.78	58.33	83.33	108.33	136.11
(50 + 50 + 400)	116.67	83.33	150.0	216.67	194.44
1:2 CHROMATIC		16.67	30.56	55.56	83.33
(67 + 133 + 300)		44.44	38.89	100.0	100.0
INTENSE CHROMATIC			25.0	50.0	77.78
(100 + 100 + 300)			66.67	136.33	111.11
SOFT DIATONIC				25.0	52.78
(100 + 150 + 250)				66.67	61.11
INTENSE DIATONIC					38.39
(100 + 200 + 200)					55.56

5-55. *Ordered combinatorial intervallic magnitude metric on Polansky (upper) and difference (lower) interval sets on permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	168.77	168.77	7.10	222.66	215.57
	450.06	450.06	18.92	229.76	215.57
28/27 · 5/4 · 36/35		9.46	171.14	161.68	168.77
		9.46	435.87	431.14	450.06
36/35 · 5/4 · 28/27			161.68	171.14	168.77
			431.14	435.87	450.06
36/35 · 28/27 · 5/4				225.03	222.66
				225.03	229.76
5/4 · 28/27 · 36/35					7.10
					18.92

5-56. *Ordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	175.0 466.67	233.33 233.33
50 + 400 + 50		175.0 466.67

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	100.0 266.67	133.33 133.33
100 + 300 + 100		100.0 266.67

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	50.0 133.33	66.67 66.67
200 + 100 + 200		50.0 133.33

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	50.0 133.33	25.0 66.67	83.33 150.0	91.67 116.67	100.0 100.0
100 + 250 + 150		75.0 200.0	33.33 33.33	75.0 200.0	83.33 150.0
150 + 100 + 250			75.0 200.0	66.67 66.67	91.67 116.67
150 + 250 + 100				75.0 200.0	50.0 133.33
250 + 100 + 150					25.0 66.67

Unordered counterparts of the ordered metrics are also defined. Although the unordered linear absolute or intervallic magnitude metrics are of little use in this context, the unordered combinatorial intervallic magnitude (UCIM) metric is rather interesting when computed on these two interval sets.

For the Polansky interval set, the metric is:

$$\left|\sum_{j=1}^{L-1}\sum_{i=1}^{L-j}\Delta(e_{1_i}, e_{1_{i+j}})/L_m - \sum_{j=1}^{L-1}\sum_{i=1}^{L-j}\Delta(e_{2i}, e_{2_{i+j}})/L_m\right|, L_m = 6.$$

This function is the absolute value of the difference between the averages of the corresponding intervals. For the difference set, the formula becomes:

$$\left|\sum_{i=2}^{L}(I_{1_i})/L_m - \sum_{i=2}^{L}(I_{2_i})/L_m\right|, L_m = 6,$$

where the I_{n_i} are the elements of the set.

5-57 and 5-58 show the data for the same group of tetrachords as before. Genera are fairly well discriminated by this metric, especially when calculated on the Polansky interval set, but not as well with the difference set intervals. Neither are particularly successful for distinguishing permutations with this metric (5-59 and 5-60).

5-57. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	11.78 47.11	10.49 44.54	16.98 73.77	25.86 119.68	19.37 106.71
28/27 · 15/14 · 6/5		1.29 2.57	5.20 26.65	14.08 72.57	7.59 59.60
25/24 · 16/15 · 6/5			6.48 29.23	15.36 75.14	8.88 62.17
22/21 · 12/11 · 7/6				8.88 45.91	2.39 32.94
16/15 · 9/8 · 10/9					6.48 12.97

5-58. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from tempered tetrachords.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC 50 + 50 + 400	13.889 61.11	8.333 50.0	16.67 83.33	25.0 116.67	19.44 116.67
1:2 CHROMATIC 67 + 133 + 300		5.556 11.11	2.778 22.22	11.11 55.56	5.556 55.56
INTENSE CHROMATIC 100 + 100 + 300			8.333 33.33	16.67 66.67	11.11 66.67
SOFT DIATONIC 100 + 150 + 250				8.333 33.33	2.778 33.33
INTENSE DIATONIC 100 + 200 + 200					5.556 0.0

5-59. *Unordered combinatorial intervallic magnitude metric on Polansky (upper) and difference (lower) interval sets on permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	56.26 225.03	56.26 220.30	2.36 4.73	2.36 117.24	0.0 107.78
28/27 · 5/4 · 36/35		0.0 4.73	53.89 220.30	53.89 107.78	56.26 117.24
36/35 · 5/4 · 28/27			53.89 215.57	53.89 103.05	56.26 112.51
36/35 · 28/27 · 5/4				0.0 112.51	222.66 103.05
5/4 · 28/27 · 36/35					2.36 9.46

5-60. *Unordered combinatorial intervallic magnitude metric on the Polansky (upper) and difference (lower) interval sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	58.33 233.33	0.0 116.67
50 + 400 + 50		58.33 116.67

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	33.33 133.33	0.0 66.67
100 + 300 + 100		33.33 66.67

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	16.67 33.33	0.0 33.33
200 + 100 + 200		16.67 66.67

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	16.67 66.67	8.333 16.67	16.67 83.33	8.333 16.67	0.0 50.0
100 + 250 + 150		25.0 83.33	0.0 16.67	25.0 50.0	16.67 16.67
150 + 100 + 250			25.0 100.0	0.0 33.33	8.333 66.67
150 + 250 + 100				25.0 66.67	16.67 33.33
250 + 100 + 150					8.333 33.33

In addition to absolute and intervallic metrics, directional metrics are also defined. Directional metrics measure only the contours of musical structures, i.e., whether the differences between successive elements are positive, negative or zero. Although these metrics are perhaps the most interesting of all, they are generally inapplicable to tetrachords because tetrachords are sets of four monotonically increasing pitches whose differences are always positive (or negative if the tetrachord is presented in descending order). Directional metrics, however, are very applicable to melodies constructed from the notes of tetrachords or from tetrachordally derived scales such as those of chapter 6.

The intervals of the tetrachordal difference set, however, are not necessarily monotonic and therefore combinatorial directional metrics may be computed on these intervals. Two such metrics were calculated for the same set of tetrachords and permutations used above, the ordered

5-61. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from tetrachords in just intonation.*

	28/27 · 15/14 · 6/5	25/24 · 16/15 · 6/5	22/21 · 12/11 · 7/6	16/15 · 9/8 · 10/9	12/11 · 11/10 · 10/9
28/27 · 36/35 · 5/4	.1667 .3333	.1667 .3333	.1667 .3333	.5000 .3333	.1667 .3333
28/27 · 15/14 · 6/5		0.0 0.0	0.0 0.0	.3333 .6667	0.0 0.0
25/24 · 16/15 · 6/5			0.0 0.0	.3333 .6667	0.0 0.0
22/21 · 12/11 · 7/6				.3333 .667	0.0 0.0
16/15 · 9/8 · 10/9					.5000 .3333

5-62. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from tempered genera.*

	1:2 CHROMATIC	INTENSE CHROMATIC	SOFT DIATONIC	INTENSE DIATONIC	EQUAL DIATONIC
ENHARMONIC (50 + 50 + 400)	.1667 .3333	0.0 0.0	.1667 .3333	.5000 .3333	.3333 .6667
1:2 CHROMATIC (67 + 133 + 300)		.1667 .3333	0.0 0.0	.3333 .6667	.5000 1.00
INTENSE CHROMATIC (100 + 100 + 300)			.1667 .3333	.5000 .3333	.3333 .6667
SOFT DIATONIC (100 + 150 + 250)				.3333 .6667	.5000 1.00
INTENSE DIATONIC (100 + 200 + 200)					.3333 .3333

combinatorial intervallic directional (OCID) metric and its unordered counterpart, the unordered combinatorial intervallic directional (UCID) metric. The OCID metric is the average of the differences of the signs of corresponding intervals. The sign (sgn) of an interval is -1, 0, or +1 according to whether the interval is decreasing, constant or increasing. The difference (diff) is 1 when the signs are dissimilar, otherwise the difference is zero. The definition of the OCID metric on the difference set is:

$$\sum_{i=2}^{L} \mathrm{diff}(\mathrm{sgn}(I_{1i}), \mathrm{sgn}(I_{2i}))/L_m,\ L_m = 6.$$

The UCID metric is the average of the absolute values of the numbers of intervals with each sign. The definition of UCID on the difference set is:

$$\sum_{i=2}^{L} \mid \#e_1^v - \#e_2^v \mid)/L_m, L_m = 6,$$

where $\#e_n^v$ = the number of intervals in the matrix such that $v = \mathrm{sgn}(I_{ni})$; i.e., $v = [-1, 0, 1]$.

The data from these computations are shown in 5-61 and 5-62. Similar results were obtained with tetrachordal permutations (5-63 and 5-64).

5-63. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from permutations of Archytas's enharmonic genus.*

	28/27 · 5/4 · 36/35	36/35 · 5/4 · 28/27	36/35 · 28/27 · 5/4	5/4 · 28/27 · 36/35	5/4 · 36/35 · 28/27
28/27 · 36/35 · 5/4	.5000	.5000	.1667	.1667	0.0
	.3333	.3333	.3333	.3333	0.0
28/27 · 5/4 · 36/35		0.0	.3333	.3333	.5000
		0.0	.6667	0.0	.3333
36/35 · 5/4 · 28/27			.3333	.3333	.5000
			.6667	0.0	.3333
36/35 · 28/27 · 5/4				.3333	.1667
				.6667	.3333
5/4 · 28/27 · 36/35					.1667
					.3333

5-64. *Ordered (upper) and unordered (lower) combinatorial interval direction metrics on difference sets from permuted tempered tetrachords.*

ENHARMONIC	50 + 400 + 50	400 + 50 + 50
50 + 50 + 400	.5000 .6667	.3333 .3333
50 + 400 + 50		.5000 .3333

INTENSE CHROMATIC	100 + 300 + 100	300 + 100 + 100
100 + 100 + 300	.5000 .6667	.3333 .3333
100 + 300 + 100		.5000 .3333

INTENSE DIATONIC	200 + 100 + 200	200 + 200 + 100
100 + 200 + 200	.5000 .3333	.3333 .3333
200 + 100 + 200		.5000 .6667

SOFT DIATONIC	100 + 250 + 150	150 + 100 + 250	150 + 250 + 100	250 + 100 + 150	250 + 150 + 100
100 + 150 + 250	.3333 .6667	.1667 .3333	.3333 .6667	.1667 .3333	.3333 .6667
100 + 250 + 150		.5000 .3333	0.0 0.0	.5000 .3333	.3333 0.0
150 + 100 + 250			.5000 .3333	0.0 0.0	.1667 .3333
150 + 250 + 100				.5000 .3333	.3333 0.0
250 + 100 + 150					.1667 .3333

Rothenberg propriety

David Rothenberg has developed criteria derived from the application of concepts from artificial intelligence to the perception of pitch (Rothenberg 1969, 1975, 1978; Chalmers 1975, 1986b). In Rothenberg's own words (personal communication): "These concepts relate the intervallic structure of scales to the perceptibility of various musical relations in music using these scales. Only the relative sizes of the intervals between scale tones, not the precise sizes of these intervals are pertinent." These concepts are applicable to scales of any cardinality whether or not the intervals repeat at some interval of equivalence. In practice, most scales repeat at the octave, though cycles of tetrachords and pentachords are found in Greek Orthodox liturgical music (Xenakis 1971; Savas 1965).

To apply Rothenberg's concepts, the first step is to construct a difference matrix from the successive intervals of an *n*-tone scale. The columns of the matrix are the intervals measured from each note to every other one of the scale. The rows t_n of the matrix are the sets of adjacent intervals measured from successive tones. These intervals are defined conventionally: the row of seconds (t_1) comprises the differences between adjacent notes; the row of thirds (t_2) consists of the differences between every other note; etc., up to the interval of equivalence (t_n). Row t_0 contains the original scale.

A number of functions may be calculated on this matrix. The most basic of these is *propriety*. A scale is *strictly proper* if for all rows every interval in row t_{n-1} is less than every interval in row t_n. If the largest interval in any row t_{n-1} is at most equal to the smallest interval in row t_n, the scale is termed *proper*. These equal intervals are considered ambiguous as their perception depends upon their context. A familiar example is the tritone (F–B in the C major mode in 12-tone equal temperament), which may be perceived as either a fourth or a fifth.

Scales with overlapping interval classes, i.e., those with intervals in rows t_{n-1} larger than those in rows t_n, are *improper*. These contradictory intervals tend to confound one's perception of the scale as a musical entity, and improper scales tend to be perceived as collections of principal and ornamental tones. Improper scales may contain ambiguous intervals as well.

5-65 illustrates these concepts with certain tetrachordal heptatonic scales in the 12- and 24-tone equal temperaments. The first example is the intense diatonic of Aristoxenos. The scale is proper and the tritone is ambiguous. The second scale is Aristoxenos's soft diatonic which is also

5-65. Rothenberg difference matrices. The row index is t_n. Max (t_n) is the largest entry in row t_n. Min (t_n) is the smallest enry in row t_n. The intense diatonic tetrachord is 1 + 2 + 2 degrees or 6 + 12 + 12 parts. The soft diatonic derives from 2 + 3 + 5 or 6 + 9 + 15 parts. The neutral diatonic is 3 + 4 + 3 degrees, a permutation of 9 + 9 + 12 parts. The intense chromatic is 1 + 1 + 3 degrees. The enharmonic tetrachord is 1 + 1 + 8 degrees. Intervals in parentheses are ambiguous; those in square brackets are contradictory.

proper, but replete with ambiguous intervals. A composer using this scale might prefer to fix the tonic with drone or restrict modulation so as to avoid exposing the ambiguous intervals. The next scale is patterned after certain common Islamic scales employing modally neutral intervals. It is strictly proper, a feature it shares with the more familiar five-note black key scale in 12-tone equal temperament.

The final two examples, Aristoxenos's intense chromatic and his enharmonic, are improper. The majority of the intervals of these scales are either ambiguous or contradictory. These scales are most likely to be heard and used as pentatonic sets with alternate tones or inflections.

Because the major (0 400 700 cents, 4:5:6 in just intonation), minor (0 300 700 cents, 10:12:15), subminor (0 250 700 cents, 6:7:9), and supramajor (0 450 700 cents, 14:18:21) triads are strictly proper, they can serve

INTENSE DIATONIC IN 12-TONE ET: PROPER

t_0	0	2	4	6	7	9	11	12/0	
t_1	1	2	2	2	1	2	2		MAX (t_3) = MIN (t_4) = 6
t_2	3	4	4	3	3	4	3		
t_3	5	(6)	5	5	5	5	5		
t_4	7	7	7	7	(6)	7	7		
t_5	8	9	9	8	8	9	9		
t_6	10	11	10	10	10	11	10		
t_7	12	12	12	12	12	12	12		

SOFT DIATONIC IN 24-TONE ET: PROPER

t_0	0	2	5	10	14	16	19	24/0	
t_1	2	3	(5)	4	2	3	(5)		MAX (t_1) = MIN (t_2)
t_2	(5)	8	(9)	6	(5)	8	(5)		MAX (t_2) = MIN (t_3)
t_3	10	(12)	11	(9)	10	10	10		MAX (t_3) = MIN (t_4)
t_4	14	14	14	14	(12)	13	(15)		MAX (t_4) = MIN (t_5)
t_5	16	17	(19)	16	(15)	18	(19)		ETC.
t_6	(19)	22	21	(19)	20	22	21		
t_7	24	24	24	24	24	24	24		

NEUTRAL DIATONIC IN 24-TONE ET: STRICTLY PROPER

t_0	0	3	7	10	14	17	21	24/0	
t_1	3	4	3	4	3	4	3		MAX (t_{n-1}) < MIN (t_n)
t_2	7	7	7	7	7	7	6		
t_3	10	11	10	11	10	10	10		
t_4	14	14	14	14	13	14	13		
t_5	17	18	17	17	17	17	17		
t_6	21	21	20	21	20	21	20		
t_7	24	24	24	24	24	24	24		

INTENSE CHROMATIC IN 12-TONE ET: IMPROPER

t_0	0	1	2	5	7	8	9	12/0	
t_1	1	1	[3]	[2]	1	1	[3]		MAX (t_1) > MIN (t_2)
t_2	[2]	4	[5]	3	[2]	4	4		MAX (t_2) > MIN (t_3)
t_3	5	(6)	(6)	[4]	5	5	5		
t_4	7	7	7	7	(6)	(6)	[8]		
t_5	8	8	10	8	[7]	9	10		
t_6	9	11	11	9	10	11	11		
t_7	12	12	12	12	12	12	12		

ENHARMONIC IN 24-TONE ET: IMPROPER

t_0	0	1	2	10	14	15	16	24/0	
t_1	1	1	[8]	4	1	1	[8]		MAX (t_1) > MIN (t_2)
t_2	[2]	9	[12]	5	[2]	9	9		MAX (t_2) > MIN (t_3)
t_3	10	[13]	[13]	[6]	10	10	10		MAX (t_3) > MIN (t_4)
t_4	14	14	14	14	[11]	[11]	[18]		
t_5	15	15	22	15	[12]	19	22		
t_6	16	23	23	16	20	23	23		
t_7	24	24	24	24	24	24	24		

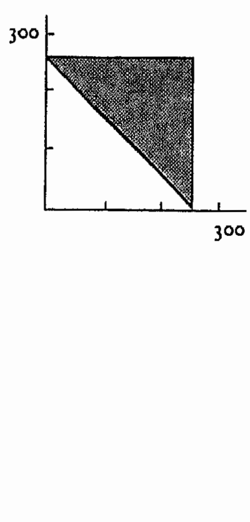
5-66. Propriety limits of tetrachords. The differences are in cents and an underlying zero modulo 12 equal temperament is assumed. The results for just intonation are virtually identical except that the fourth of 498.045 cents and a whole tone of 203.91 cents replace the 500- and 200-cent intervals in the computations.

ROWS	DIFFERENCE MATRIX		
t_1	a	b	$500 - a - b$
t_2	$a + b$	$500 - a$	$500 - b$
t_3	500	500	500

CONSTRAINTS: $0 < a < 250$; $0 < b < 250$; $250 < a + b < 500$.

VERTICES: 0, 250; 250, 0; 250, 250.

5-67. Propriety limits for isolated tetrachords and conjunct chains of tetrachords.



as sets of principal tones for improper scales. The various sets of principal tones would be used as the main carriers of melodies, while the auxiliary tones would be used as ornaments. This topic deserves more extended discussion than is appropriate here and Rothenberg's original papers should be consulted (Rothenberg 1969, 1975, 1978).

The fact that the minor and septimal minor triads are strictly proper may explain certain musically significant cadential formulae in the Dorian modes of the enharmonic and chromatic genera. These consist of a downward leap from the octave to the lowered submediant (trite), then down to the subdominant (mese) before ending up on the dominant (paramese). This formula may be repeated a fifth lower, beginning with a leap from the subdominant (mese) to the lowered supertonic (parhypate) and then down to the *subtonic* (hyperhypate) before ending on hypate (chapters 6 and 7). Minor triads are outlined in the chromatic genus and septimal minor triads in the enharmonic. The latter chords contain the important interval of five dieses called eklysis by the Greek theorists, and in fact, the jump from parhypate to hyperhypate is seen in the *Orestes* fragment (Winnington-Ingram 1936). The upper submediants (lichanos and paranete) may be substituted in both genera; the major triad appearing in the chromatic genus is also strictly proper.

As has been seen above, the propriety criterion separates those scales derived from chromatic and enharmonic tetrachords from those generated by diatonic genera. As will be seen later, the situation is somewhat more complex; under certain conditions, some diatonic tetrachords yield only improper scales, while some chromatic genera can combine with diatonic tetrachords to generate proper mixed heptatonic scales.

Propriety may be computed for abstract classes of scales or subscalar modules rather than for specific instances by replacing one or more of the intervals by variables. If the three subintervals of the tetrachord are written as a, b, and $500 - a - b$ (a, b, and $4b/3a$ in just intonation), one can calculate the Rothenberg difference matrix and determine the propriety limits for isolated tetrachords or conjunct chains where the interval of equivalence is the fourth. Such chains were present in the earlier stages of classical Greek music and are still extant in contemporary Greek Orthodox liturgical music (chapter 6 and Xenakis 1971).

The computation is performed by solving the inequalities formed by setting each of the elements of rows t_n less than each of those in rows t_{n+1}.

In practice, the work may be minimized because only the elements in the first $(n + 1) / 2$ rows of an n-tone scale need be considered. One may also ignore relations that are tautological when all the intervals are positive.

The result is a set of constraints on the sizes of intervals a and b, shown in 5-66. Tetrachords and conjunct chains of tetrachords spanning perfect fourths, are strictly proper when intervals a and b satisfy these constraints. The tetrachords and chains are proper when their intervals equal the extrema of the constraints. For values outside these limits, the tetrachords and conjunct chains are improper.

Because the three intervals a, b, and $500 - a - b$ add to a constant value, there are only two degrees of freedom. Therefore, the domain over which tetrachords are proper may be displayed graphically in two dimensions. The region in the $a \cdot b$ plane within which tetrachords are strictly proper is shown in 5-67. The vertices define an area in the $a \cdot b$ plane within which the constraints are satisfied. Points on the edges of the triangular region correspond to proper tetrachords. The two points on the axes are also proper as *trichords*, which are degenerate tetrachords with only three notes.

Similarly, the propriety limits for pentachords consisting of a tetrachord and an annexed disjunctive tone (200 cents or 9/8) may be determined. The difference matrix is shown in 5-68. As all circular permutations of a scale have the same value for propriety, it is immaterial whether the disjunctive tone is added at the top or bottom of the tetrachord. The region satisfying the propriety constraints for isolated pentachords and pentachordal chains is shown in 5-68.

Similar calculations may be carried out for complete heptatonic scales consisting of two identical tetrachords and a disjunctive tone. This tone

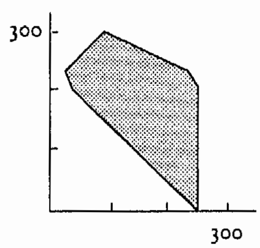
5-68. *Propriety limits of pentachords.*

ROWS	DIFFERENCE MATRIX			
t_1	a	b	$500 - a - b$	200
t_2	$a + b$	$500 - a$	$700 - a - b$	$200 + a$
t_3	500	$700 - a$	$700 - b$	$200 + a + b$
t_4	700	700	700	700

CONSTRAINTS: $0 < a < 250$; $0 < b < 250$; $250 < a + b < 500$; $2a + b < 700$; $a + 2b < 700$; $b - a < 200$; $300 < 2a + b$.

VERTICES: 250, 0; 50, 200; 33.3, 233.3; 100, 300; 233.3, 233.3; 250, 200.

5-69. *Propriety limits for isolated pentachords and conjunct chains of pentachords.*



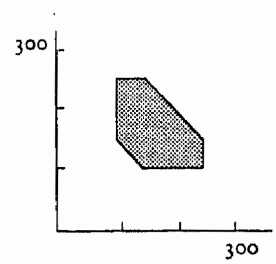
5-70. *Propriety limits for heptatonic scales with identical tetrachords.*

a	b	$500-a-b$	200	a	b	$500-a-b$
$a+b$	$500-a$	$700-a-b$	$200+a$	$a+b$	$500-a$	$500-b$
500	$700-a$	$700-b$	$200+a+b$	500	500	500
700	700	700	700	$500+a$	$500+b$	$1000-a-b$

CONSTRAINTS: $100 < a < 250$; $100 < b < 250$; $250 < a + b < 400$.

VERTICES: 100,150; 100,250; 150,100; 150,250; 250,150; 250,100.

5-71. *Propriety limits for heptatonic scales with identical tetrachords.*



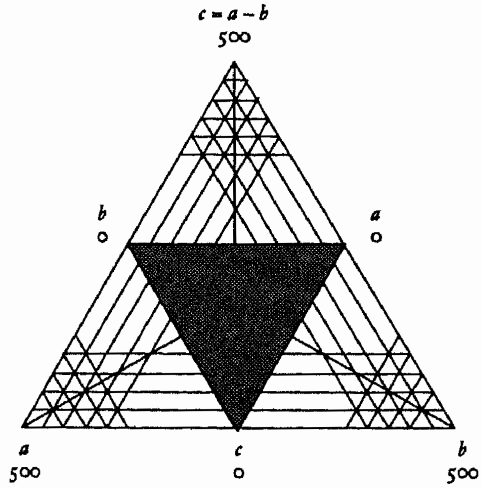
may be placed between the tetrachords or at either end to complete the octave (chapter 6). The results of the calculations are given in 5-70. The region of propriety is shown in 5-71.

Complete tetrachordal space

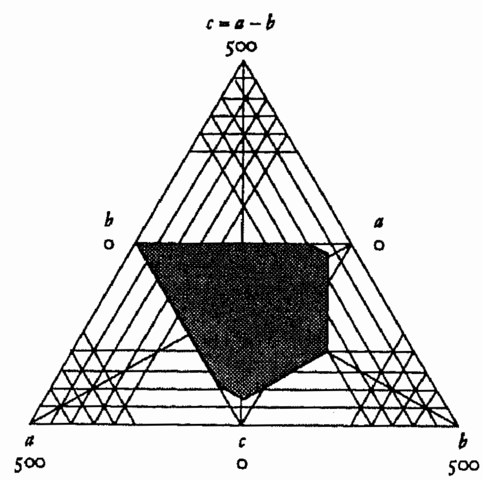
An alternative mode of graphic representation may be clearer. Physical chemists have long been accustomed to plotting phase diagrams for three component mixtures on equilateral triangle graphs. The three altitudes are interpreted as the fractions of each component in the whole mixture. There are only two degrees of freedom as the sum of the composition fractions must equal unity. The data from 5-66, 5-68, and 5-70 have been replotted in 5-72–73.

5-72 shows the range over which the intervals *a*, *b*, and 500 − *a* − *b* may vary and still result in proper tetrachords. Pentachords are shown in 5-73 and heptatonic scales in 5-74.

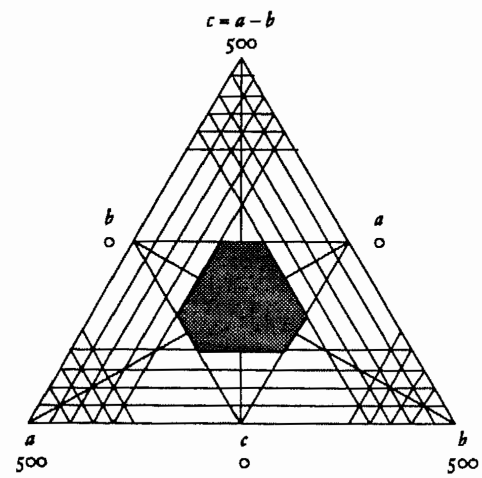
The advantage of the triangular graph over the conventional rectangular type is most evident with the heptatonic scales of 5-74. All points in the interior of the semi-regular hexagonal region correspond to strictly proper scales, while the edges are sets of intervals that define scales that are merely



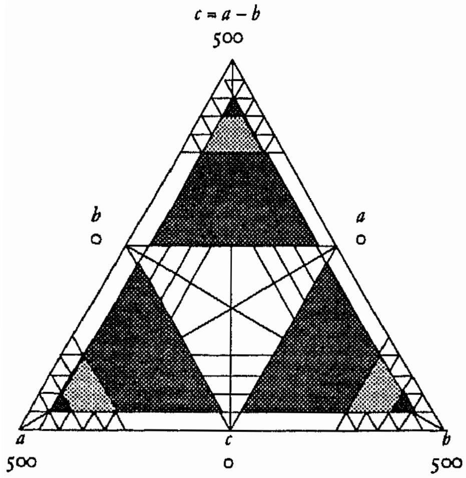
5-72. *Propriety limits for tetrachords and tetrachordal chains. These limits are for chains of conjunct tetrachords such as are found in Greek Orthodox liturgical music (Xenakis 1971).*



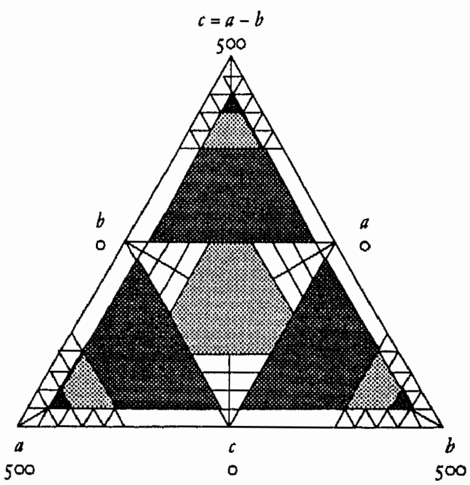
5-73. *Propriety limits for pentachords and pentachordal chains.*



5-74. *Proper heptatonic scales.*



5-75. *Non-diatonic genera.*



5-76. *Complete tetrachordal space.*

proper. The three triangular spaces lying between the long sides of the hexagon and the edge of the space contain diatonic genera which yield improper heptatonic scales. In certain cases to be discussed later, some of these tetrachords may be combined with other genera to produce proper mixed scales.

The six vertices of the central hexagon in 5-74 are the six permutations of the soft diatonic genus of Aristoxenos, 100 + 150 + 250 cents. The center of overall symmetry is the equal diatonic genus, 166.667 + 166.667 + 166.667 cents. The intersection of the altitudes of the triangle and the midpoints of the long sides of the hexagon are the three permutations of the intense diatonic, 100 + 200 + 200 cents, while the intersections with the midpoints of the short sides define the arrangements of the neo-Aristoxenian genus, 125 + 125 + 250 cents. This genus lies on the border of the chromatic and diatonic genera, but sounds chromatic because of the equal division of the pyknon.

The non-diatonic or pyknotic genera are portrayed in 5-75. The empty border around the filled regions delimits the commatic (25 cents) and subcommatic intervals. The small triangular regions in dark color near the vertices are the hyperenharmonic genera whose smallest intervals fall between 25 and 50 cents in this classification (see the neo-Aristoxenian classification above for more refined limits on the boundaries between the hyperenharmonic, enharmonic, and chromatic genera). Next are the trapezoidal enharmonic and chromatic zones which flank the unmarked central diatonic area. The enharmonic zone contains pyknotic intervals from 50 to 100 cents and the chromatic from 100 to 125 cents.

These data are summarized in 5-76. The diatonic tetrachords generating proper and strictly proper scales map into the central zone. The three triangular zones flanking the central region along the long sides of the hexagon are diatonic tetrachords which contain one of the small hyperenharmonic, enharmonic, or chromatic intervals. These diatonic genera yield improper scales. As in 5-75, the chromatic tetrachords lie in the large trapezoidal regions, with the enharmonic and hyperenharmonic beyond. The outer belts of the chromatic zones depict genera with enharmonic and hyperenharmonic intervals. Similarly, the enharmonic regions are divided into realms of pure enharmonic and enharmonic mixed with hyperenharmonic intervals.

Propriety of mixed scales

The computation of the propriety limits for heptatonic scales containing dissimilar tetrachords is a more complex problem. Since there are now four degrees of freedom, two for each of the tetrachords, the graphical methods used for the single tetrachord case are of limited use. It is possible, however, to consider the upper and lower tetrachords separately and to calculate absolute limits on the intervals of each. If a, b, and $500 - a - b$ are assigned to the intervals of the lower tetrachord and c, d, and $500 - c - d$ to the upper, one can compute the range of values for a and b over which it is possible to find an upper tetrachord with which a proper scale can be generated. Similar computations may be done for c and d. These results of these calculations are tabulated in 5-77 and are graphed in 5-78 and 5-79. These graphs use only those relations which are solely functions of a and b or c and d.

Triangular plots of the same data are depicted in 5-80 and 5-81. The union of the the upper and lower tetrachord regions corresponds to the pentachordal limits of 5-68 and 5-73, and their intersection is the proper diatonic region of 5-74. The upper and lower tetrachord regions are also the intervallic retrogrades of each other as propriety is unaffected by retrogression or circular permutation of the intervals.

The solution to the general case of finding the limits for mixed tetrachordal scales must satisfy all the inequalities that relate a, b, c, and d. It is difficult to display this four-dimensional solution space in two dimensions. One can, however, choose tetrachords from the lower or upper absolute

5-77. Propriety limits for heptatonic scales with mixed tetrachords. (Only the first four rows are shown.)

a	b	$500 - a - b$	200	c	d	$500 - c - d$
$a + b$	$500 - a$	$700 - a - b$	$200 + c$	$c + d$	$500 - c$	$500 - c - d + a$
500	$700 - a$	$700 - a - b + c$	$200 + c + d$	500	$500 - c + a$	$500 - c - d + a + b$
700	$700 - a + c$	$700 - a - b + c + d$	700	$500 + a$	$500 - c + a + b$	$1000 - c - d$

CONSTRAINTS ON a AND b: $0 < a < 250$; $250 < a + b < 500$; $2a + b < 700$; $a + 2b < 700$.

VERTICES: *100, 150; 100, 300; 250, 200; 250, 0; 233.3, 233.3.*

CONSTRAINTS ON c AND d: $c < 250$; $250 < c + d < 400$; $d - c < 200$; $300 < 2c + d$.

VERTICES: *50, 200; 33.3, 233.3; 100, 300; 250, 150; 250, 0.*

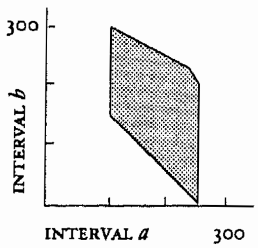
MUTUAL CONSTRAINTS ON a, b, c, AND d: $a < c + d$; $b < c + d$; $c < a + b$; $d < a + b$; $c < 2a$; $a + c < 500$; $b + c < 500$; $a + d < 500$; $b - c < 200$; $2c - a < 300$; $a - c < 100$; $c + d - a < 300$; $a + b + c < 700$; $2c + d - a < 500$; $c + 2d - a < 500$; $a + b + d < 700$; $2a + 2b - c < 700$; $a + b - c - d < 100$; $300 < a + c + d$; $c + d < 2a + b$; $200 < 2a + 2b - c - d$; $2c + d - a - b < 300$; $2a - c - d < 500$; $200 < 2a + b - c$; $c + b + d - a < 500$; $500 < a + b + c + d$; $300 < 2c + 2d - a$; $2a + b - 2c - d < 200$.

propriety regions of 5-80 and 5-81 and find companion tetrachords which produce proper heptatonic scales when joined to them by a disjunctive tone. These computations are performed in the same way as in 5-70 and 5-77, except that the variables in one of the two tetrachords are replaced by the cents values of the intervals. The result of the calculations will be a range of values for the companion tetrachord.

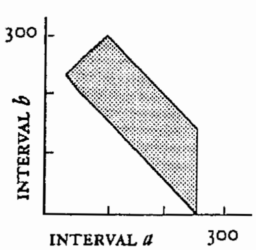
The three permutations of the intense diatonic genus in 12-tone equal temperament (100 + 200 + 200 cents, 200 + 100 + 200 cents, and 200 + 200 + 100 cents) as well as the neochromatic form of the syntonic chromatic (100 + 300 + 100 cents) were selected as lower tetrachords. The propriety limits for the upper companion tetrachords were then computed. These results are shown in 5-82.

Points in the interiors of the regions yield strictly proper scales, while those on the peripheries produce scales that are merely proper. The neochromatic tetrachord has only a one-dimensional solution space; the uppermost point corresponds to a mode of the harmonic minor scale.

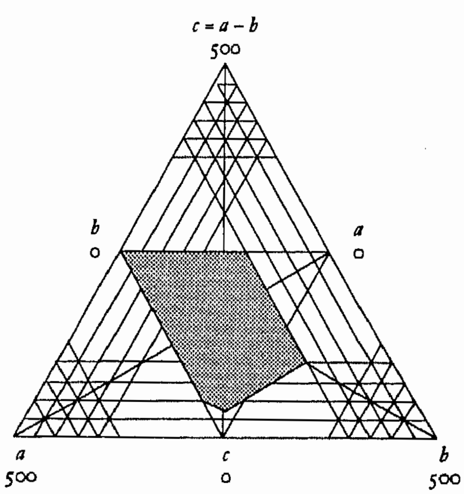
Similar calculations were performed for an additional 23 tetrachords and the results are tabulated in 5-83. In agreement with previous results (5-74 and 5-78), no proper scales could be formed from lower tetrachords whose first intervals were microtones.



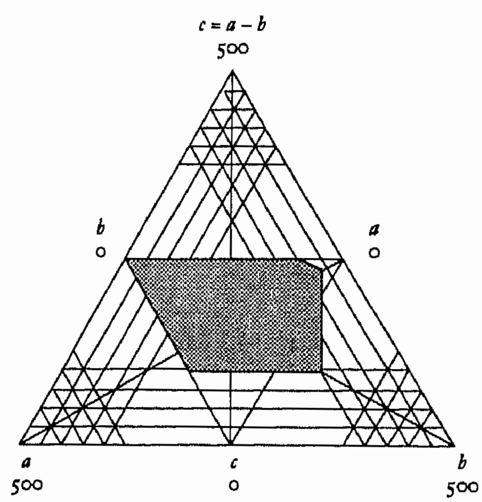
5-78. *Absolute propriety limits for lower tetrachords.*



5-79. *Absolute propriety limits for upper tetrachords.*

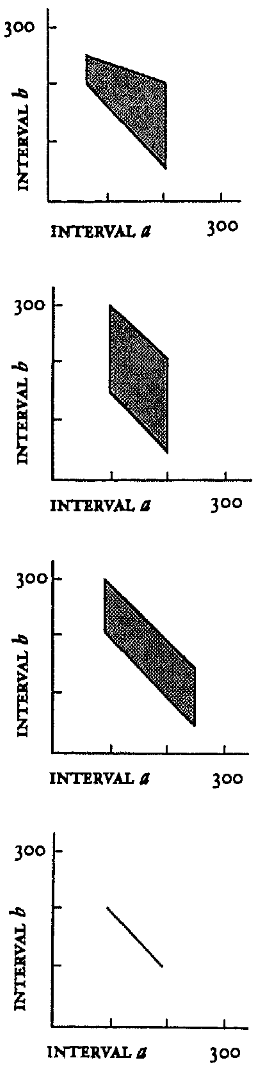


5-80. *Absolute propriety limits for lower tetrachords.*



5-81. *Absolute propriety limits for upper tetrachords.*

5-82. *Propriety ranges for upper companion tetrachords: limits for the tetrachords (a) 100 + 200 + 200 cents, (b) 200 + 100 + 200 cents, (c) 200 + 200 + 100 cents, (d) 100 + 300 + 100 cents..*



Upper tetrachords may also be chosen and lower companion ranges subsequently calculated to yield scales that are the intervallic retrogrades or octave inversions of those above.

A number of interesting conclusions may be drawn from these data. Proper heptatonic tetrachordal scales containing microtones are only possible under certain conditions. The microtonal intervals may be present in either the upper or lower tetrachord provided they are not in the extreme positions, i.e., not intervals *a* or $500-c-d$.

Proper hexatonic scales also exist when tetrachordal intervals *b* or *d* equal zero and *a* and *c* are 250 cents. These scales may be analysed as containing a tetrachord, a disjunctive tone, and a trichord.

The tetrachordal genera which appear as vertices of the propriety regions are of great interest. In particular, the equal division 166.667 + 166.667 + 166.667 accepts as upper companions both chromatic and improper diatonic genera, including some with subcommatic intervals. Other new tetrachords occurring as vertices are the improper diatonic genera 33.333 + 233.333 + 233.333; this is very close to Al-Farabi's 49/48 · 8/7 · 8/7, and 50 + 250 + 200, which is approximated rather well by 40/39 · 52/45 · 9/8.

Work of other investigators

Several other investigators have independently developed descriptors functionally identical to Rothenberg's strict propriety. Gerald Balzano has used the notion of "coherence" in his work on microtonal analogs of the diatonic scale in 12-tone equal temperament (Balzano 1980). Though not tetrachordal, Balzano's scales are homologous to the tritriadic scales discussed in chapter 7. Ervin Wilson (personal communication) has applied the term *constant structure* to scales in which each instance of a given interval subtends the same number of subintervals, but not necessarily subintervals of the same magnitude or order. This property is also equivalent to propriety.

5-83. *Proper mixed tetrachord scales, in cents. These tetrachords can combine with a disjunctive tone and any tetrachord in the region defined by the vertices to yield proper or strictly proper scales. The retrogrades of these tetrachords can also serve as the upper tetrachords of proper scales. The third interval of each tetrachord may be found by subtracting the sum of the two tabulated intervals from 500 cents. The neochromatic tetrachord number 4 is the upper tetrachord of the harmonic minor mode. Its region of propriety is reduced to a line rather than an area in the tetrachordal interval plane. Tetrachords 11, 12, and 26 cannot form proper scales with any upper tetrachord.*

	LOWER TETRACHORD	VERTICES
1.	100 200 200	50, 200; 50, 250; 200, 200; 200, 50
2.	200 100 200	100, 150; 100, 300; 200, 200; 200, 50
3.	200 200 100	100, 200; 100, 300; 250, 150; 250, 50
4.	100 300 100	100, 200; 200, 100
5.	100 150 250	50, 250; 50, 200; 150, 150; 150, 100
6.	100 250 150	100, 150; 100, 250; 200, 150; 200, 50
7.	150 100 250	50, 200; 50, 250; 150, 150; 150, 100
8.	150 250 100	100, 275; 100, 200; 150, 250; 225, 175; 225, 75
9.	250 100 150	150, 150; 150, 250; 250, 150; 250, 50
10.	250 150 100	150, 150; 150, 250; 250, 150; 250, 50
11.	50 250 200	NO PROPER SCALES
12.	50 200 250	NO PROPER SCALES
13.	200 50 250	100, 150; 100, 200; 150, 150; 150, 100
14.	200 250 50	200, 150; 200, 200; 250, 150; 250, 100
15.	250 50 200	150, 150; 150, 250; 200, 200; 200, 100
16.	250 200 50	200, 150; 200, 200; 250, 150; 250, 100
17.	125 125 250	50, 200; 50, 250; 150, 150; 150, 100
18.	125 250 125	87.5, 187.5; 87.5, 287.5; 212.5, 162.5; 212.5, 62.5
19.	250 125 125	150, 150; 150, 250; 250, 150; 250, 50
20.	150 150 200	50, 200; 50, 250; 200, 200; 200, 50
21.	150 200 150	75, 175; 75, 225; 83.3, 283.3; 150, 250; 225, 175; 225, 25
22.	200 150 150	100, 150; 100, 300; 250, 150; 250, 0
23.	100 275 125	87.5, 187, 5; 87.5, 237.5; 200, 125; 200, 75
24.	125 275 100	100, 175; 100, 250; 212.5, 137.5; 212.5, 62.5
25.	233.33 233.33 33.33	233.33, 133.33; 233.33, 166.67
26.	33.33 233.33 233.33	NO PROPER SCALES
27.	166.7 166.7 166.7	66.67, 183.33; 66.67, 266.67; 88.89, 288.89; 133.33, 266.67; 233.33, 166.67; 233.33, 16.67