A MATHEMATICAL MODEL FOR OPTIMAL TUNING SYSTEMS

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IN THIS PAPER WE PROPOSE a mathematical framework for the optimization of tuning systems. We begin with an informal definition of "tuning system." We then propose five general constraints that seem common to their evolution. The central idea of this paper is the quantification of those constraints in terms of a set of numerical parameters. Given a choice of parameter values we use appropriate optimization methods to produce an optimal tuning for a specific set of values. Finally, we consider some historical and Javanese tunings from this perspective, and use the framework to generate a few examples of novel tuning systems.

Tuning systems

A tuning system is a set of intonations for intervals or pitch classes. A tuning system might be used by a musical culture, group of musicians, or even a single composer. Such a system may also serve as an abstraction, or model, for the derivation of any number of related systems and sub-systems.

Smaller, functional subsets of pitches, such as scales, are extracted from a larger tuning system for specific musical purposes. Examples include the formation of major and minor (and other) scales from 12-tone equal temperament (12-ET), or the various Javanese pathet (*manyura*, *nem*, *sanga*, etc.) which are subsets of *slendro* and *pelog* tuning (Perlman, 40–43).

Tuning systems are neither static nor rigid. Although most musical cultures need some agreed-upon standard for musicians to tune their instruments and sing to, tuning systems evolve and fluctuate over time and in space (i.e., historically and geographically) and vary stylistically within musical practice. Most musical cultures have some standard or canonical tuning, articulated in either oral or in written traditions. Such a system may often be canonized in a specific instrument, like the piano, or the gendér in Central Javanese music which may hold the tuning for an entire gamelan.

We are interested here in a formal framework for tuning systems themselves, not the intricate (but no less important) musical variations and manifestations of such a system. Musicians deviate freely and artistically from standardized tunings in fluid, complex ways. For example, the many musical genres that share the nomenclature and intervallic template of 12-ET (like jazz and blues) are intonationally diverse. But the complexities and nuances of intonational usage associated with established tuning systems are beyond the scope of this paper.

Culturally- and historically-specific constraints may influence the formation of tuning systems. A new system that resembles a pre-existing one is often desirable, as in Central Javanese gamelan tunings which reference well-known gamelans.[1] A tuning system might adapt over time in the performance of an evolving body of music. This latter consideration is an important factor in the historical evolution of tunings in European music over the past millennium, including just intonations (JIs), meantones, well-temperaments (WTs), equal temperaments (ETs), and twentieth-century experimental tuning systems.

From a formal, abstract perspective tuning systems can be seen as specific attempts to solve certain problems, and understood as the resolution of a particular set of intonational constraints. The genesis of these problems—whether they emanate from issues of culture, economy, convenience, aesthetics, or some complex combination of all of these—is another issue. Our focus is on a relatively small set of important factors common to the creation of tuning systems, whose natural quantitative formulation enables the use of *optimization* techniques for analysis. We believe this approach has important implications towards a deeper understanding of tuning and even musical style.

Five Constraints

Tuning systems through history and across cultures are the result of a set of complex compromises aimed at simultaneously incorporating some or all of the following structural constraints:

1. *Pitch set*: use of a fixed number of pitches (and consequently, a fixed number of intervals);

2. *Repeat factor*: use of a *modulus*, or *repeat factor*[2] for scales, and for the tuning system itself (e.g., an octave);

3. *Intervals*: an idea or set of ideas of correct or "ideal" intervals, defined in terms of frequency relationships;

4. *Hierarchy*: a ranking of importance for the accuracy of intervals with respect to the ideal intervals;

5. *Key*: a hierarchy of importance of specific scales begun at various pitches in the system.

Most tuning systems attempt to resolve some or all of the five constraints listed above. The "best" fit may also reflect and incorporate other theoretical, cultural, historical, and even aesthetic factors. However, these five fundamental structural constraints, which appear to operate at a different, less culturally specific level, can be stated formally and mathematically. In this formulation, there is an associated *weighting system* that allows the generation of any tuning system with a fixed number of pitches, repeat-factor, and some set of ideal intervals. Not all tuning systems, of course, consider all of these constraints in their construction. Neither are they exhaustive.[3] However, these constraints constitute an economical and musically reasonable set capable of providing an interesting analysis of any tuning system.

Comments on the Five Constraints

Some concept of a *number of pitches* seems to be nearly universal, primarily, perhaps, for practical reasons.[4] In actual performance practice the situation is usually more complex. Style-specific ideas of "between-ness" and intonational variation are sometimes well-articulated in theory and pedagogy, yet it is difficult to completely account for intra- and intercultural variation. For that reason, most theoretical tuning literature has focused on canonical tuning systems. We adopt a similar approach here, assuming a fixed number of "discrete" (Burns and Ward, 243) pitches for a tuning system.

While musics vary widely in the numbers of basic and/or named pitches in use, having a fixed number of pitches offers practical and cognitive advantages. The most obvious benefit is economical, as it affects the construction of instruments. There are pedagogical and musical advantages to a finite, even moderately sized set of pitches from which to learn to compose, improvise, sing, and perform common repertoires (Dowling and Harwood, 92; Burns and Ward, 244). Some authors (e.g., Lerdahl) have proposed cognitive explanations for certain numbers of pitches in a system. Whatever the specific optimal numbers may be, cultures seem to eventually agree on a specific number of pitches for a tuning system.[5]

Repetition of an interval set at some fixed interval, or *modulus*, is common, if not universal, and is related to *having* a fixed set of pitches. Both music theory and cognition often distinguish between *pitch height* and *pitch chroma* (Shepard 1964; 1982), the latter usually referring to pitches whose frequencies are related as powers of two, or octaves. Although the modulus interval is usually the octave (or something close to it), this is not always the case, as in the well-known Pierce-Boehlen scale (Matthews and Pierce; see also, Moreno), or the stretched octaves of Central Javanese tuning (Surjodiningrat et al.; Sethares; Polansky 1984). The universality of the octave is not generally disputed, but the extent to which its prevalence is best accounted for by hard-wired or learned cognitive mechanisms is uncertain (e.g., Burns and Ward, 262–264, and for a discussion of octave equivalence in other primates, see Hauser and McDermott). Regardless, most tuning systems seem to *repeat*, or "cycle," (Dowling and Harwood, 19) at some fixed interval. This interval, in practice, can be somewhat flexible, incorporating spectral considerations (such as the stretched octaves of piano tuning), musical factors, or some combination of the two. Given the utility of a fixed set of pitches, a reasonably constant repeat-factor is a likely corollary.[6] In our framework we do not assume a specific interval of repetition—only that the system repeats at a constant interval.[7]

Tuning systems that place a greater importance on some intervals than others are called hierarchical (Krumhansl 1990; in particular, Chapter 10). The mathematical simplicity of an interval is often correlated with its importance (such as a "pure" fifth, octave, or third). There are cognitive (Trehub) and acoustical arguments for the importance of these intervals—they are easily heard, measured, and produced, and are likely candidates to function as generators for larger sets of intervals. As a consequence, the mistuning of such "important" intervals may be of greater concern. Even in 12-ET, this hierarchy exists: the octave is exactly 2/1, the fifth is extremely close to 3/2 (2 cents difference), and the third much further from 5/4 (14 cents difference).

Subsets of tuning systems (scales, modes, *pathet*, or *rags*, etc.) may similarly demonstrate such a hierarchy. Many, if not most, bluegrass songs, for example, seem to be in open keys on the guitar, mandolin, and fiddle (G, D, A, E), and it is common for bluegrass guitarists to tune to an "open G chord," privileging its tuning slightly over 12-ET. In other words, it is more important that the key of G be "in tune" than the key of C♯. Historically, European art music has implemented these kinds of relationships in mean-tone tunings and well-temperaments. Key hierarchies also occur in the *pathet* (roughly: *scales*) of Central Javanese gamelan music. The tuning system called slendro has three pathet (*manyura*, *nem*, *sanga*), which have complex usage relationships implying intonational constraints for particular melodic patterns and intervals (Perlman, 44–45; Sumarsam, 142–143).

It is important to state *a priori* that the mathematical formulation outlined here does not assume that tuning systems are based on small, rational intervals. Indeed, many tuning systems (such as slendro and 12-ET) deviate significantly from those kinds of intervals. Nor do we mean to engage longstanding discussions of "categorical perception" (Burns and Ward, 250–254; Patel, 24–26; Sethares, 50) or "consonance" (Tenney 1988). The purpose of this framework is to consider tuning systems through the analysis of stylistically non-specific variables such as number of pitches, repeat factor, ideal intervals, interval and key weights.

Historical Example: Well-Temperament

Examination of a few well-known constructs from historical European tuning theory can illustrate some of the motivations for the mathematical framework described below. Several concepts central to our framework are introduced here: the *interval matrix*, *ideal intervals*, the associated *error matrix*, and the *error function*.

Table 1 shows a common Just Diatonic scale (begun arbitrarily on "C").

1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	(ratios)
C	D	E	F	G	A	B	C	(note names)
0	204	386	498	702	884	1088	1200	(cents)

TABLE 1: JUST DIATONIC SCALE. CENTS VALUES ARE FOR INTERVALS IN RELATION TO 1/1 ("C").

By definition, JIs are "in tune" in one key (possibly a few others) but problematic in others. Certain intervals that represent differences between a fixed number of scale degrees will differ from some *ideal interval* which appears elsewhere in the scale. In the Just Diatonic scale in Table 1, the most important such interval (aside from the 2/1 octave) is often considered to be the 3/2 perfect fifth. The musical fifth between the second and sixth degrees of the scale above (40/27, the well known "wolf-fifth") differs from that between the first and fifth or the third and seventh scale degrees.[8] Central to this phenomenon is that intervals in this scale are built on more than one prime: in this case, the primes 2, 3 and 5. A tuning system such as JI can't have pure intervals involving different primes and still be in tune with itself. The wolf-fifth is only one example, if perhaps the most famous.[9] An even simpler version of this phenomenon occurs in the well-known Pythagorean comma,[10] which involves only the primes 2 and 3.

This fundamental problem, which can be called the *collision of primes*, has no simple resolution, and has motivated the development of many tuning systems. It is also, in part, a consequence of the constraints of a repeating tuning with a fixed number of notes. WTs, and ultimately, equal-temperaments (ETs), approximate ideal intervals in more complex ways, moving towards the equality of derived scales beginning on different notes.

We might refer to this as the *historical tuning problem*,[11] a musical/mathematical ramification of the insolvability of the Diophantine equation $p^n = q^m$ for distinct primes p and q (and $n, m > 0$).[12] The problem for music theory is quite general, and might more appropriately be referred to not by its species name ("wolf"), but by its family name: the "canidae interval," more generally describing any interval that is the result of a collision of primes and therefore necessitates some sort of compromise in the tuning system itself.[13,14] But this "collision" will occur in any system with irregular intervals—not just rational intervals. Thus, any system (except for ET) that has some set of ideal intervals, fixed set of pitches, and repeat factor needs to develop systematic compromises in order to resolve this problem.

The *interval matrix* is a useful way to encode a tuning system. The *i,j* (row, column) entry in this matrix is the interval between notes *i* and *j*. Table 2 shows the half-matrix[15] of the JI diatonic scale[16] clearly showing the wolf fifth at entry (D, A) or (2, 6):

	C (1/1)	D (9/8)	E (5/4)	F (4/3)	G (3/2)	A (5/3)	B (15/8)	C (2/1)
C		9/8	5/4	4/3	**3/2**	5/3	15/8	2/1
D			10/9	32/27	4/3	**40/27**	5/3	16/9
E				16/15	6/5	4/3	**3/2**	8/5
F					9/8	5/4	45/32	**3/2**
G						10/9	5/4	4/3
A							9/8	6/5
B								16/15

TABLE 2: INTERVAL HALF-MATRIX OF JUST DIATONIC SCALE. INTERVALS ARE IN RATIOS WITHIN ONE OCTAVE. FIFTHS ARE IN BOLD. THE WOLF (40/27) IS THE ONLY NON-IDEAL PERFECT FIFTH. NOTE THAT ALL MAJOR THIRDS (C–E, F–A, G–B) ARE IDEAL INTERVALS OF 5/4

An *ideal tuning* would be one in which the *i,j* entry only depends on $|i-j|$—each entry is equal to an ideal interval. In the *ideal interval matrix*, values on the diagonals are constant and equal to the ideal ratio. The ideal interval matrix is equivalent to the interval matrix only in ET. Any unequal interval propagates itself through the matrix, causing irregularity.

The entries of the *error matrix* of a tuning system are the differences between the entries of the interval matrix and the respective entries in the *ideal* interval matrix. Tables 3 and 4 show respectively the interval matrix and a subset of the error matrix for one of the most studied WTs in history: the so-called "Werckmeister III" (W3) tuning, devised by Andreas Werckmeister, an influential Seventeenth-Century music theorist and keyboard tuner.[17]

	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
C		*90*	*192*	*294*	*390*	*498*	*588*	*696*	*792*	*888*	*996*	*1092*	*1200*
C#			102	204	300	408	498	606	702	798	906	1002	1110
D				102	198	306	396	504	600	696	804	900	1008
Eb					96	204	294	402	498	594	702	798	906
E						108	198	306	402	498	606	702	810
F							90	198	294	390	498	594	702
F#								108	204	300	408	504	612
G									96	192	300	396	504
G#										96	204	300	408
A											108	204	312
Bb												96	204
B													108

TABLE 3: W3 HALF-MATRIX. EACH DIAGONAL IS A SPECIFIC INTERVAL WITH VALUES GIVEN IN CENTS. "KEYS" CORRESPOND TO ROWS. ALL KEYS IN W3 ARE CONSIDERED REASONABLY GOOD.

A comparison of these half-matrices shows that in a WT system like W3 the degree of error varies considerably over intervals and keys (shown here as diagonals and rows). Table 5 focuses on the error distribution of a few of the intervals. Smaller errors in W3 tend to be found in central keys (beginning on the first, fifth, and seventh degrees) and in important intervals like the major third and perfect fifth.

All WTs (and W3 in particular) illustrate the technique we describe below. They are also an excellent, clearly articulated set of examples representing the idea that tuning systems seem to evolve as attempts to reconcile ideas of key, ideal interval, a fixed number of pitches, and something like the octave, into an *optimal* tuning system. Well-documented by the Eighteenth-Century theorists who developed them, WTs have been discussed and analyzed a great deal ever since (Barbour, Donahue, Jorgensen, Lindley).

	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
C					4	0		6					
C#						22	0		0				
D							10	6		6			
Eb								16	0		0		
E									16	0		0	
F										4	0		0
F#		0									22	6	
G			6									10	6
G#		0		0									22
A		16	6		0								
Bb			10	0		0							
B				16	0		6						

TABLE 4: W3 ERROR HALF-MATRIX. THREE (DIAGONAL) INTERVALS (MAJOR THIRD, PERFECT FOURTH, PERFECT FIFTH) ARE SHOWN FOR THE HALF-MATRIX. ERRORS ARE COMPUTED RELATIVE TO IDEAL INTERVALS OF 386, 498, AND 702 CENTS FOR THE THREE INTERVALS.[18]

Key	C	C#	D	E♭	E	F	F#	G	G#	A	B♭	B
M3	4	22	10	16	16	4	22	10	22	16	10	16
P4	0	0	6	0	0	0	6	6	0	6	0	0
P5	6	0	6	0	0	0	0	6	0	0	0	6
Error (cents)	10	22	22	16	16	4	28	22	22	22	10	22

TABLE 5: KEYS IN W3. USING THE CENTS VALUES IN FIGURE 4 FOR THE MAJOR THIRD, PERFECT FOURTH, PERFECT FIFTH, THE ERROR DISTRIBUTION FOR W3 (IN CENTS, FROM ONE SPECIFIC SET OF INTERVALS) CAN BE SEEN, SHOWING AN ALMOST SYMMETRICAL INCREASE (VIA THE CIRCLE OF FIFTHS) AROUND THE "CENTRAL" OR "BEST" KEY OF F.

Mathematical Formulation

A rigorous notion of an *optimal tuning system* may be formulated by providing a formal, mathematical framework for some seemingly universal and important aspects of tuning. We begin by formalizing the five constraints on tuning systems described above. Next, we use this language to define a general notion of the *interval matrix* and the associated *error matrix* as well as an *error function* that uses the error matrix as input. We then minimize this error function using least squares to find the optimal tuning.

The five tuning system constraints are formalized as follows:

1. Pitch set: let $a_1, \ldots, a_n$ be a set of n pitches, none equal to 0.

2. Repeat factor: let $\omega > a_n$ be the *repeat factor* of the tuning system.

3. Intervals: let $I_1, \ldots, I_n$ represent the *ideal intervals.*

4. Hierarchy: let $i_1, \ldots, i_n$ be *interval weights* used to represent the desired accuracy of the n intervals in the tuning system.

5. Key: let $k_0, \ldots, k_n$ be *key weights* used to represent the fixed pitches in the tuning system from which intervals are measured.

Using this notation, the interval matrix M for a set of n pitches, $a_1, \ldots, a_n$, is written as

$$M = \begin{pmatrix} 0 & a_1 & \cdots & a_{n-1} & a_n \\ \omega - a_1 & 0 & \cdots & a_{n-1} - a_1 & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega - a_{n-1} & \omega + a_1 - a_{n-1} & \cdots & 0 & a_n - a_{n-1} \\ \omega - a_n & \omega + a_1 - a_n & \cdots & \omega + a_{n-1} - a_n & 0 \end{pmatrix}.$$

The matrix M has $n+1$ rows and $n+1$ columns with zeros along the diagonal. The entry in row i and column j is $a_j - a_i$ if $i \le j$, or $\omega + a_j - a_i$ if $i > j$. The matrix M specifies all possible intervals of the tuning system. All instances of a particular interval are found on the diagonals of the matrix.

The ideal interval matrix L represents the desired interval for each entry in the matrix M:

$$L = \begin{pmatrix} 0 & I_1 & \cdots & I_{n-1} & I_n \\ I_n & 0 & \cdots & I_{n-2} & I_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_2 & I_3 & \cdots & 0 & I_1 \\ I_1 & I_2 & \cdots & I_n & 0 \end{pmatrix}.$$

Note that L is circulant (i.e., is generated by rotating each row to the right relative to the previous row).

The error matrix is the difference between the interval matrix M and the ideal interval matrix L. The total error of a tuning system is defined as a function of the error matrix. Natural choices for this function include the sum of the absolute values of the entries, known as the L^1 error, or the square root of the sum of the squares of the entries, known as the L^2 error.[19] We use the L^2 error, which at least mathematically, is a natural choice, giving the Euclidean distance between the ideal and interval matrices. It also has the advantage of producing a simple solution to the optimization problem (the so-called "method of least squares") that is achieved via differentiation. However, we can, and do, also investigate the use of other functions for which solutions can be found via a genetic-algorithm approach that we have developed.[20]

In the absence of any key or interval hierarchy (i.e., the interval weights and key weights are all equal) the L^2 error function is:

$$E(\vec{a}) = \sum_{i,j} \left(M_{i,j} - L_{i,j}\right)^2,$$

where the vector $\vec{a}$ contains the n pitches, a_1 to a_n, and the subscripts (i, j) denotes the element of the associated matrix at row i and column j. In Appendix A, we prove that ET is always the optimal solution for this case, independent of the ideal intervals specified in the matrix L. This result is in accord with the historical evolution of tuning systems: if no key or interval is preferred over any other key or interval, ET is the best solution, since by definition it has no intonational differences between keys. If, however, as in WTs, some keys or intervals are more important than others, then ET may not be the best solution in the least-squares sense.

To formalize the notion of the relative importance of intervals and keys, we use interval and key weights. These weights can be applied to the error function through a weight matrix W:

$$W = I * K$$

where the $*$ operator denotes an element-wise product (i.e. $W_{i,j} = I_{i,j}K_{i,j}$) and the interval-weight matrix I and key-weight matrix K are defined as

$$I = \begin{pmatrix} 0 & i_1 & \cdots & i_{n-1} & i_n \\ i_n & 0 & \cdots & i_{n-2} & i_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ i_2 & i_3 & \cdots & 0 & i_1 \\ i_1 & i_2 & \cdots & i_n & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_0 & k_0 & \cdots & k_0 & k_0 \\ k_1 & k_1 & \cdots & k_1 & k_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-1} & k_{n-1} & \cdots & k_{n-1} & k_{n-1} \\ k_n & k_n & \cdots & k_n & k_n \end{pmatrix}.$$

The weighted version of the error function is therefore

$$\hat{E}(\vec{a}) = \sum_{i,j} W_{i,j}\left(M_{i,j} - L_{i,j}\right)^2$$

or

$$\hat{E}(\vec{a}) = \sum_{i=0}^{n} k_i \left[\sum_{j=0}^{i-1} i_{n+j-i+1}(\omega + a_j - a_i - I_{n+j-i+1})^2 + \sum_{j=i+1}^{n} i_{j-i}(a_j - a_i - I_{j-i})^2 \right]$$

For each set of constraints specified in the matrices W and L, there is a unique solution that minimizes the weighted error function (see Appendix A).[21] This solution is called the *optimal tuning system*, and it is a set of pitches $a_1, \ldots, a_n$. While the optimal tuning is unique to a given set of constraints, the converse is not true: there is not necessarily a unique set of constraints that will generate a given tuning. In other words, multiple sets of constraints can generate the same tuning, within a specified tolerance.[22]

For example, Table 6 shows the interval and key weights used to approximate W3 to within an average error of 0.5 cents by using one (Pythagorean) interval (3/2) to generate a set of ideal intervals.[23]

Though many sets of constraints can generate W3, these *particular* constraints illustrate an interesting alternative view of the tuning: W3 can be generated by specifying good perfect fifths in the *outer keys*.[24] This is surprising in light of common formulations of WTs in which "inner" keys are considered to be central to their design. In other words, the idea of tuning to central keys may be historically true but not mathematically necessary.[25]

Ideal intervals	0	90	204	294	408	498	612	702	792	906	996	1110
Int. weights		-	-	-	-	-	-	1	-	-	-	-
Key weights	0.3	-	0.15	0.05	0.05	0.1	-	0.2	0.05	0.1	-	-

TABLE 6: A GOOD APPROXIMATION OF W3. KEY AND INTERVAL WEIGHTS DERIVED FROM THE INTERVAL 3/2 GENERATE AN APPROXIMATION OF W3 WITH AN AVERAGE ERROR OF 0.5 CENTS.

Using this framework, we can, with a high degree of accuracy, design and produce tuning systems incorporating historical and cross-cultural criteria. With some high degree of accuracy, we can approximate criteria for the development of historical tuning systems. Similarly, by heuristically replicating a tuning system, we can infer something about the criteria important to its design, such as the key and interval weights that would produce it from a given set of ideal intervals (or vice versa). This approach can be taken a step further, and used to invent entirely new systems based on arbitrary criteria.

This framework specifies tuning systems parameterically, using a formal description of higher level "features" (five constraints) rather than "note-by-note" intervals. A given set of features uniquely describes one tuning system. This can be an interesting and powerful approach to the study of scales, tuning systems, and tuning in general, as well as a creative tool for new compositional ideas.

Examples: Optimal Well Temperaments

A convenient point of comparison for our model is W3, since it is an historical version of the method proposed here. ". . . [W3] was the first unequal temperament to allow satisfactory performance of all possible tonalities" (Ledbetter, 38). Using the information contained in the interval and error matrix, we can rigorously analyze and expand upon statements such as the above description of W3, as well as other tunings. In some sense, all tuning systems other than ETs are well-tempered: they must attempt to resolve some or all of these constraints. From this perspective, our framework might be alternatively described as a way of creating *optimal well-temperaments* (OWTs).

EXAMPLE 1: OWT1 AND OWT2 AND MINIMAL MEAN-TEMPERING

As a way of exploring the possibilities of our framework to formally generalize historical criteria for tuning systems, we created several new WTs. Rasch, Chalmers (1974), and others have proposed simple, reasonable measurements of tuning systems.[26] Rasch, in his consideration of Werckmeister's tunings, measures the *mean-tempering* of "all consonant intervals, which is equal to the mean tempering of all triads, or of all keys" (Rasch, 38–46). This is the mean difference of the three intervals in the major chord tuned as 1/1, 5/4, 3/2 (ideal intervals) and the actual intervals in the tuning over twelve keys. Rasch's measure is thus an *error function* for a tuning system given a set of ideal intervals, distinct from, but related to the one used in this paper.[27] Rasch uses this function to measure the degree to which Werckmeister's tuning systems are "in tune," evaluating the temperament in terms of its pure major thirds and perfect fifths (and consequently, major triads). In addition, Rasch's measure provides us with a meaningful, comparitive way to measure the results of experiments in generating new WTs.

W3 is exemplary in its mean-tempering of 10.43 cents, which is the same as 12-ET (and can be shown to be an absolute minimum[28]). An historical WT often known as Young 2 (sometimes considered to be an improvement on W3 (Jorgensen; Donahue)), also achieves this minimality.

Using our framework, we generated two new optimal tunings with the same minimal mean-tempering as W3, Young 2, and 12-ET. Using reasonable sets of ideal intervals, we found sets of weights[29] for two new optimal tunings (OWT1, OWT2), each of which is maximally in tune by a specific measure: mean-tempering of triads. OWT1 and OWT2 have a great deal in common (ideal intervals, key and interval weights), theoretically and musically, with historical WTs. However, their musical implications and structure differ in important ways from their historical models. Table 7 shows all four tunings (W3, Young 2, OWT1, and OWT2).

The tempering of major triads in W3, Young 2, and the two generated tunings OWT1 and OWT2, is shown in Figure 1. The tunings can be viewed as four different distributions of the minimal error, each with different characteristics. Both W3 and OWT2 have certain triads that are more in tune than any in Young 2 or OWT1. OWT2 contains two adjacent triads with equal minima, while W3 has only one minimal triad. This means that where W3 has one best key, OWT2 has two. OWT1 has two non-contiguous minima areas. For both OWT1 and OWT2 having several minima means that some other triads will be less in tune. This is historically unusual, and might suggest a more chromatic musical style. Young 2 never achieves the triadic minima of W3 or OWT2, but has a wide adjacent region of relatively in-tune triads. It is important to reiterate that OWT1, OWT2, W3, and Young 2 all have the same minimum total mean-tempering error.

W3	0	90.2	192.2	294.1	390.2	498.0	588.3	696.1	792.2	888.3	996.1	1092.2
Young 2	0	90.2	196.1	294.1	392.2	498.0	588.3	698.0	792.2	894.1	996.1	1090.2
OWT1	0	102.0	203.8	297.2	396.3	498.1	600.0	702.0	803.8	897.2	996.3	1098.1
OWT2	0	93.1	203.1	296.3	397.4	498.5	591.7	701.6	794.8	903.4	997.4	1091.4

TABLE 7: FOUR DIFFERENT MINIMALLY-TEMPERED WTS (TWO HISTORIC, TWO SYNTHETIC). YOUNG 2 AND W3 ARE IMPORTANT HISTORICAL TUNINGS; OWT1 AND OWT2 ARE GENERATED BY OUR FRAMEWORK, AND LIKE W3 AND YOUNG 2, ARE "MINIMALLY MEAN-TEMPERED" BY RASCH'S MEASURE.

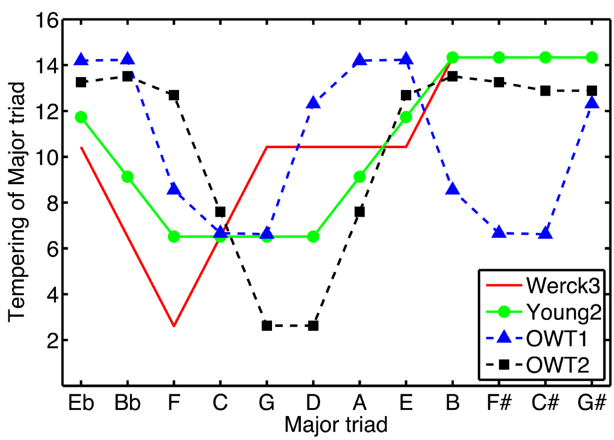

FIGURE 1: TEMPERING OF TRIADS IN FOUR MINIMALLY-TEMPERED WTs. EACH OF THE FOUR "MINIMALLY TEMPERED" WTS (TWO HISTORIC, TWO SYNTHETIC) HAS A DIFFERENT STRUCTURE OF THE KEY-ERROR DISTRIBUTION. NOTE THAT W3 IS THE ONLY ONE OF THE FOUR WITH A SINGLE "BEST" KEY, WHILE OWT2 HAS TWO "BEST" KEYS THAT ARE AS GOOD AS W3'S "WINNER."

Our framework can generate new tunings that belong to a kind of *minimality class* (along with W3 and Young 2) but, at the same time, explore different musical possibilities. Furthermore, with our optimality method, we can represent those tunings parametrically as features (weights, ideal intervals, etc.) corresponding to the constraints that shape the development of historical WTs.

EXAMPLE 2: SEPTIMAL OWTS

Of course this framework can be used to create new and experimental optimal tunings, and in doing so, it is interesting and fun to work out various speculative investigations. For example, what if Werckmeister had listened to Bach in the morning and Lightnin' Hopkins at night, and become fascinated with the flat minor thirds and minor sevenths which might suggest the septimal intervals 7/6 and 7/4 respectively? Septimal intervals have fascinated many composers, including Partch, Fokker, Harrison and others, and the commonality of septimal intervals (particularly the minor third and minor seventh) has been frequently conjectured (if difficult to empirically substantiate).[30] Table 8 shows the result of another experiment in the creation of new tunings in which we add septimal minor thirds and minor sevenths to the ideal interval set while keeping major thirds and perfect fifths pure.

We recorded several Preludes and Fugues from the *Well Tempered Clavier* in these new septimal tunings, and in OWT1 and OWT2 (along with, for comparison, the conventional W3 and Young 2) (Polansky 2007).

Many of the intervals in these septimal OWTs are unusual, such as the quarter-tone minor seconds and the flat minor thirds and minor sevenths. However, the usual fifths and thirds are fairly well maintained. The resulting tunings, heard in the context of such a familiar musical work, will seem strange at first. Yet these tunings are, in some sense, a simple, transparent extension of the fundamental premises of WT, incorporating a new prime in the ideal interval set. They give us some idea of how this framework generates new scales according to specific, demonstrable criteria (constraints). We like to think that both Bach and Werckmeister might have enjoyed these excursions into new tonalities, and perhaps even appreciated and understood their underlying ideas.

SEPTIMAL OWT1[31]

Ideal intervals		100	204	267	386	498	600	702	800	900	969	1100	
Int. weights		.0001	0.001	0.3	1	1	0.001	1	.001	0.001	1	0.001	
Key weights	1	0.001	0.001	0.3	0.001	1	0.001	1	0.001	0.001	0.001	0.001	
Resultant	0	42	206	272	386	491	543	704	764	877	977	1090	1200

SEPTIMAL OWT2

Ideal intervals		100	204	267	386	498	600	702	800	900	969	1100	
Int. weights		0.1	0.001	0.3	1	1	0.001	1	.001	0.001	1	0.001	
Key Weights	1	0.001	0.001	0.001	0.001	1	0.001	1	0.001	0.001	0.001	0.001	
Resultant	0	93	209	262	386	490	588	707	769	876	978	1093	1200

TABLE 8: "SEPTIMAL OWTS." TWO SPECULATIVE OWTS WHICH INTEGRATE SEPTIMAL INTERVALS, ILLUSTRATING THE USE OF THE FRAMEWORK TO GENERATE NEW AND EXPERIMENTAL TUNING SYSTEMS.

Pilot Study: The "concept" of slendro

As a further example of how our mathematical framework might be used as a means to explore "tuning space," we consider a tuning system from another, radically different, musical culture: Central Javanese *slendro*. Slendro is a five-tone *laras* or "tuning," one of the two main interlocking scales in Central Javanese classical music, or *karawitan*. Gamelans generally have two sets of instruments, one tuned in slendro, the other in seven-tone *pelog*. Slendro and pelog usually share a common pitch. In general, no two gamelans are tuned alike. However, as mentioned earlier, there are well-known, influential tunings, and gamelans are, for various reasons, sometimes tuned specifically to match another's tuning (Brinner, 52).

However, gamelan musicians and scholars often discuss the qualities of different slendro tunings, and it is common for some slendros to be considered "better" than others, as well as more appropriate for one or more of the three main *pathet* (roughly: mode) that are used. There is a clear and strong concept of "slendro," or what Brinner (53) calls a "perceptual category," and Perlman and Krumhansl (112) call a "perceptual set."

Because of its unusual combination of variability and strength as a concept, slendro has long fascinated musicians of many styles and genres (the composer Lou Harrison called it "slippery"). Brinner states: "While the 'spacing' of [the] pitches varies from one set of instruments to the next, it is not random. . ." (53). Yet it is difficult to find, either within the sparse literature or the voluminous anecdotal evidence, a kind of "definition" for slendro that would determine which 5-tone, more or less equally spaced, stretched-octave scale is slendro, and which is not.

There are, however, certain commonalities to documented slendro tunings. Gamelans generally use stretched octaves, and although various ranges are given in the literature, the stretch seems to fall roughly between 5–25 cents. Octave stretch also varies according to register, instrument, and even regional style. There is also a modal (pathet) hierarchy. The three common pathets are named *sanga*, *manyura* and *nem*. Each emphasizes certain pitches or scale degrees, thought to be "more important" than others in certain ways (often in terms of cadence). However, gamelan musicians do not always agree on which tones are most important, or how (in a manner similar to discussions of European tuning systems).

Sumarsam, in his discussion of Sindusawarno's mid-Twentieth-Century theoretical work on karawitan describes a "hierarchy of the functions of the tones in each pathet system." According to Sindusawarno: in pathet sanga, the three most important pitches are, in descending order, 1, 5, 2[32]; in manyura, 3, 1, 5[33]; in nem, 5, 2, 1. "The relationship of the tones is one *kempyung* [a distance of 4, more or less equivalent to a perfect fifth]. In that way, Sindusawarno neatly associates kempyung as a function of laras and kempyung as a determinant of the tones in each pathet."[34] As in European tuning systems, there is some "hierarchy of the functions of the tones [in each pathet . . .]" (Sumarsam, 143) The specifics of these tones, hierarchies, and nature of their importance is the subject of much discussion, and beyond the scope of this paper.[35]

Thus, slendro contains a *repeat factor*, *modal hierarchies*, and significant intervals which presumably are accounted for in a given tuning. These are essentially the factors which determine our mathematical formalization of temperament. In slendro tuning, much as in that of a European well-temperament, repeat-factor, important "keys," and significant interval relationships must be taken into account. Certain relationships have greater importance than others, suggesting (while not necessarily proving), that they be tuned more "precisely" in the temperament.

Musically, of course there is very little common ground between Baroque keyboard music and karawitan. The underlying tuning systems, however, seem to be conceptually more similar: the *idea of slendro*, like the *idea of 12-tone WT* supports a wide variety of tunings, all used for the same repertoire. Interestingly, slendro retains an important characteristic lost in European tunings when WTs became ETs: each pathet in slendro, unlike "keys" in modern 12-ET, have markedly different intervallic structures. Manyura *sounds* different than sanga, in a way that is different from the difference between, say, D major and F major.

The classic "Gadjah Madah" (GM) study (Surjodiningrat et al.) is often cited as a source of data for the consideration of gamelan tunings, containing careful measurements of twenty-eight gamelans from Jogyakarta and Surakarta ("Jogya" and "Solo"). The advantage of this data is that it is published, empirical (as opposed to anecdotal), and reasonably accurate. Perlman and Krumhansl report that in Perlman's own "ongoing" study of Javanese gamelan tunings, he has found "similar variability" to the GM data.[36] It is difficult to measure gamelan tunings: instruments change their pitch over time; gamelans are in various states of repair; the selection of register and instrument makes a difference; finally, the time-variant pitch envelope of bronze keys makes it difficult to decide exactly *when* to measure the fundamental. While several published individual gamelan tuning measurements exist, the GM study is especially useful as a dataset for experimentation using various analytic techniques.

Another challenge with slendro analysis is the paucity of pitches: five. This seems to create an intrinsic constraint, perhaps emanating from the predominantly linear and melodic style, resulting in a relatively even intervallic spread. All slendros are *strictly proper* by Rothenberg's definition, or what Balzano calls *coherent*. Consequently, adjacent intervals tend to cluster around the 5-ET 240 cents, perhaps more easily distinguished as being "small and large" seconds (Polansky 1984), the largest of which are more appropriately described as minor thirds. The maximum adjacent interval (second) in the GM study[37] is 266 cents (almost exactly a septimal minor third = 7/6 = 267 cents) and, the minimum is 216 cents. In other words, slendros cluster tightly around 5-ET. In the GM study the mean octave stretch is 1212 cents (with a variance of about 11 cents). The mean and variance of the seconds (rounded to the nearest tenth of a cent) are shown in Table 9.

To demonstrate how our framework might be used on a pre-existing set of data (the GM study), we ran two pilot experiments. In the first, we fixed a set of interval weights and a minimal set of ideal intervals, and searched for the best-fitting key weights. In the second, we fixed key weights and a larger set of ideal intervals (5), and searched for the best fitting interval weights.[38]

Interval (2^{nd})	1→2	2→3	3→5	5→6	6-1′
GM mean	232.8	238.8	245.5	243.6	251.6
GM variance	9.4	11.3	12.1	8.9	9.4

TABLE 9: MEAN AND VARIANCES OF ADJACENT SECONDS (1→2, 2→3, ETC.) FOR GAMELANS IN THE GM STUDY. SCALE DEGREES IN SLENDRO ARE NUMBERED 1, 2, 3, 5, 6. 1' IS THE "OCTAVE" OF 1.

EXPERIMENT 1: FITTING A SET OF SLENDROS TO TWO IDEAL INTERVALS AND VARYING INTERVAL WEIGHTS

In Experiment 1, we specified two ideal intervals: 231 cents and 702 cents for the second and fifth respectively.[39] Using those two intervals we tried four ratios of interval weights, setting all other weights to values near zero. We also did the same for "stretched" versions of these intervals in proportion to the actual octave of the tuning. In that way, a total of eight different trials were run. We used the octave stretches of each individual tuning as the repeat factor. For each trial, we searched for the best key weights, and recorded the "fit" to each of the twenty-seven GM scales, measured by the sum of the cents difference between the generated scale and the actual scale. Next we ran the same data (without the stretched intervals) on twenty-seven randomly generated slendros whose overall mean and variances were taken from the GM data.

Table 10 shows the fitting error for the GM scales. The two best fits for the randomly generated scales (1/1 = 4.10; 1/3 = 6.22, both unstretched) were significantly worse than the two best GM scales.

DISCUSSION

We should be careful when interpreting these results, which are, in fact, less interesting to us than the demonstration of a possible, more extended use of the optimization framework as a way of considering temperaments and tuning systems. That said, there are several things worth noting. First, given our assumptions, the actual scales did considerably better than the randomly generated scales, perhaps adding

2nd/5th Interval Weight Ratio	Average ¢ error
1/3	3.13
1/6	3.99
1/3 (stretched)	4.13
1/1	4.81
1/6 (stretched)	5.4
2/1	7.07
1/1 (stretched)	8.25
2/1 (stretched)	10.62

TABLE 10: AVERAGE FITTING ERROR FOR VARYING SETS OF INTERVAL WEIGHTS, USING TWO IDEAL INTERVALS (8/7, 3/2), FOR GAMELANS IN THE GM STUDY. THE WEIGHTING RATIOS TRIED ARE RANKED FROM BEST TO WORST FIT.

further evidence that there is a strong underlying structure to slendro, while not necessarily saying exactly what that structure is. Second, the unstretched intervals performed better in general better than the stretched ones, perhaps suggesting that while the octaves are stretched (perhaps to accommodate acoustic phenomenon (Sethares, 1993, 1997)), some intonational percept of a second and/or a fifth might be reasonably strong,[40] and that those intervals are tempered to the octave stretch, not vice versa (although this experiment neither proves that, nor elucidates how and when this might occur). Similar experiments using a 2/1 octave might suggest otherwise. Third, the three best fits all show a stronger influence of the fifth than the second in the tempering. Many more configurations should be tried, and with a larger dataset, to further investigate the assumption that a simple just fifth is somehow an underlying factor in slendro. It would be interesting, as well, to match the actual key weights against anecdotal evidence of how appropriate each gamelan is for the three different pathet.

EXPERIMENT 2: USING KEY WEIGHTS TO DISTINGUISH CITIES

In our second experiment, we set four ideal intervals (231, 498, 702, 996), once again opting for acoustic and/or numerical simplicity. We then ran a number of trials using different sets of key weights. Six of the most illustrative are shown in Table 11. The first trial used equal weights. The second through sixth trial set one "key" three times as high as all the others. The final trial used equal key weights for 1, 2, 5, and 6, with 3 = 0.

Key Weights	Average error (all 27 gamelans)	Ave. error (Solo)	Ave. error (Jogya)
Equal	48	35	59
Key 1 (3x)	30	24	35
Key 2 (3x)	23	19	27
Key 3 (3x)	39	29	48
Key 5 (3x)	31	23	38
Key 6 (3x)	49	34	61
Key 1=2=5=6 (3 =0)	17	17	17

TABLE 11: AVERAGE FITTING ERROR FOR GM STUDY GAMELANS, USING FIXED IDEAL INTERVALS AND VARYING KEY WEIGHTS. "3X" MEANS THAT THE "KEY" ON THE SPECIFIED PITCH WAS SET 3 TIMES HIGHER THAN ALL THE OTHERS (WHICH WERE EQUAL). THE LAST LINE OF THE TABLE SETS ALL WEIGHTS EQUAL, EXCEPT FOR THE KEY BASED ON PITCH 3, WHICH IS SET TO 0.

DISCUSSION

The GM study identifies gamelans by city (Solo and Jogya, the two predominant courts of classical Central Javanese culture). Since Solonese and Jogyanese musical styles (tunings, repertoires, techniques, etc.) are often distinguished by gamelan musicians, it seemed interesting to see if we could distinguish gamelans from different cities, if only slightly (given the sparse data) by their tunings.

Before we used our framework as a mechanism to differentiate among slendros, we tried some more obvious techniques. Simple clustering applied to the twenty-seven interval sets failed to generate a clear geometric distinction between the two cities. However, a graph of the intervals and a look at simple statistics calculated from the two cities shows that, for example, there is a great deal more variation in Solo around the "middle" of the scale (pitch 3). The maximum and minimum ranges of intervals in Solo and Jogya are {52, 36} and {38, 21} respectively. That is, the variation in GM Jogya tunings is considerably "flatter" than Solo.

While these initial experiments were interesting, not surprisingly, considering the paucity of the data, they did not enable a clear statistical grouping of the gamelans.

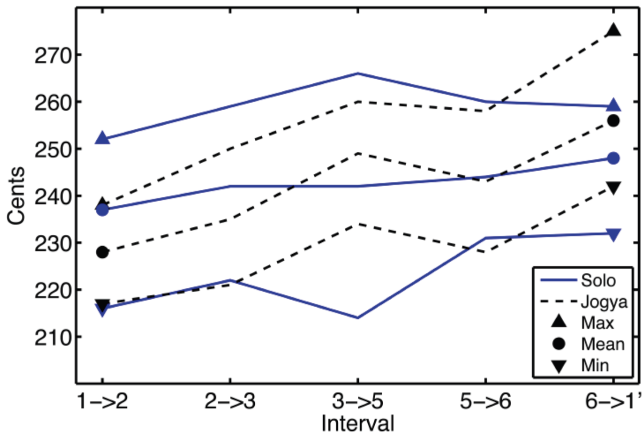

FIGURE 2: STATISTICS OF GM GAMELANS BY CITY. MEANS AND RANGES OF ADJACENT SLENDRO INTERVALS FOR JOGYA AND SOLO.

However, when we searched for interval weights after fixing key weights and ideal intervals, we found what might be significant differences in the fitting error for each category,[41] perhaps indicating more subtle tuning tendencies between the two musical styles. Two of the rows in Table 11 are especially interesting. The largest spread between the two cities occurs when note 6 receives the highest weight (although weighting them equal is close). If that fitting error is used to classify gamelans as from Jogya or Solo, we achieve a 71% classification rate, which is higher than we obtained in our clustering experiments. Recall, that in his ranking of intervals for the three pathet, Sindusawarno uniquely excludes pitch 6 as an important note (either first, second or third) from any of the three pathets. Although it is unclear how this relates to the data in Table 11 it may be due to the fact that since that pitch 6 is in some respects the least important in terms of pathet identification, giving it an unusually large value magnifies some tendency in city-specific tuning systems. It also might indicate that making the key weight on pitch 6 high "makes no sense" in any of the slendros, and generates in general, much larger fitting errors. The first line of Table 11, with equal weights, also causes a large distinction between Jogya and Solo error, and, since this is a tuning concept which is at odds with the idea of pathet differentiation via tuning, this large difference might point towards the same tendency as giving a large weight to pitch 6.

Table 11 suggests, perhaps, that given a set of simple intonational assumptions, the keys on 1, 2, and 5 predominate in the design of the tuning. These keys correspond with Sindusawarno's "winners," preferencing pathets sanga and nem over manyura. Jogya errors are greater in general, which may be a result of that city's "flatter" data in the GM study. Solo seems to favor the key built on 1 (sanga), while Jogya slightly prefers 5 (nem).

With this discussion we have tried to demonstrate a way in which this framework might be used as a new means of forming and testing hypotheses in non-western musical tuning systems. It would be of interest to fold in more data as well as a wider range of initial hypotheses in order to engage in a more comprehensive heuristic study.

Future Directions

We hope that the ideas presented here will lead to more work using these kinds of mathematical tools in the investigation of tunings. A few possibilities are included below.

1) *Further exploration of the parameter space.* Given a specified tuning, set of ideal ratios and repeat factor, there is not necessarily a unique set of corresponding weightings. Thus it is difficult to determine, from an existing tuning (whether it be a well-temperament or slendro) what the key and interval preferences might be (or have been), assuming some set of ideal intervals. It would be interesting to explore the "geometry" of the weighting space, and investigate the significance of multiple weighting solutions from a musical, historical, and/or cultural standpoint.

2) *Constraint-based system.* The interest and veracity of a constraint-based system are to a large extent dependent on the constraints. To say we can find an optimal solution for the best possible bridge from one side of the river to the other depends on what "best bridge" means. Musical problems are fuzzier than bridge-building problems. In this regard, the current framework is proposed as a model in which the set of constraints might be modified (in ways often suggested in this paper) to reflect other interests in the design of tuning systems. Adding or modifying constraints may affect the mathematical solution(s) considerably, as well as the geometry of the weighting space discussed above.

For example, a particular form of design caprice might be incorporated, such as that of desiring *one* particular interval in *one* particular key to be “just so.” In the current framework, this must be achieved by carefully setting the interval and key weights. This becomes more difficult if there are two or several such intervals. In that case, we begin to consider the modeling of tunings which aren’t quite as dependent on the notion of “abstract ideal intervals” and “what keys are most important.” There are other situations, of course, where this framework doesn’t adequately represent the social, historical, or cultural circumstances of a tuning system. We do not model factors like spiritual epiphany, or homage, or imitation, and it could be argued that these are among the most important and interesting (as well as ineffable and magical) constraints. We believe, however, that this framework accounts for *some* significant part of what makes tuning systems “tick,” and reasonably models their evolution, design, and in some sense, why some work better than others.

3) *Multiple Interval Representations.* Another useful addition to the framework (as mentioned elsewhere, in the notes) would be the possibility of having more than one ideal interval for a given position, such as the familiar situation of using *either* 81/64 or 5/4[42] for the major third. Incorporating this idea would make the mathematical representation and solution more complex, but might remedy some unnecessary awkwardness in the formulation. For example, the conventional diatonic scale has several types of seconds, thirds, etc., which result from it being a subset of a larger, chromatic system. The current framework is, by definition, much better at representing whole systems than scalar or modal subsets.

Conclusion

What are the implications of this framework for the consideration of the notion of a *tuning system* itself? Clearly, there are many issues involved in the development of a tuning system besides the specific criteria used here. A mathematical model such as ours does not account for cultural, aesthetic, historical, economic, or any number of intangible factors that might play a role in the development of a musical tradition.

Nevertheless, it might be proposed that *all* tuning systems, except for those that use rational intervals exclusively, are in fact, tempered. If that is true, then the method described here offers more than a historical analysis of the work of Werckmeister and his colleagues. The framework becomes, by extension, a general theory of tuning systems.

Acknowledgments

Thanks to Tim Polashek, who helped formulate some of these ideas in a graduate seminar at Dartmouth College long ago; and to Chris Langmead, Jody Diamond, Peter Kostelec, and Dennis Healy for valuable advice in this project.

Appendix A

In this appendix, we explore some of the mathematical properties of our framework. In particular, we show that equal temperament is the optimal tuning system if all the weights are equal and we discuss the conditions for the uniqueness of our solution.

Our weighted error function is:

$$E(\vec{a}) = \sum_{i=0}^{n} k_i \left[\sum_{j=0}^{i-1} i_{n+j-i+1}(\omega + a_j - a_i - I_{n+j-i+1})^2 + \sum_{j=i+1}^{n} i_{j-i}(a_j - a_i - I_{j-i})^2 \right]$$

First, we assume that all the weights are equal and derive the optimal tuning system in this case. A necessary condition at the minimum of the error function is that all the partial derivatives are zero. Differentiating with respect to a_k and simplifying yields:

$$\begin{aligned}
\frac{\partial E}{\partial a_k} &= \sum_{i=k+1}^{n}(\omega + a_k - a_i - I_{n-i+k+1}) + \sum_{i=0}^{k-1}(a_k - a_i - I_{k-i}) \\
&\quad - \sum_{j=0}^{k-1}(\omega + a_j - a_k - I_{n+j-k+1}) - \sum_{j=k+1}^{n}(a_j - a_k - I_{j-k}) \\
&= 2a_k + 2na_k + (n-2k)\omega - 2\sum_{j=0}^{n} a_j \\
&= (n-2k)\omega + 2(n+1)a_k - 2\sum_{j=1}^{n} a_j
\end{aligned}$$

By setting these equations equal to zero for all k, we get a system of equations that is linear in the unknowns:

$$\begin{pmatrix} 2n & -2 & \cdots & -2 \\ -2 & 2n & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & 2n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} -\omega(n-2) \\ -\omega(n-4) \\ \vdots \\ -\omega(n-2n) \end{pmatrix}$$

To simplify notation, we will write this linear system as $Ax=b$. The matrix A is full rank and is therefore invertible. The inverse of the matrix A is:

$$A^{-1} = \begin{pmatrix} \frac{1}{n+1} & \frac{1}{2(n+1)} & \cdots & \frac{1}{2(n+1)} \\ \frac{1}{2(n+1)} & \frac{1}{n+1} & \cdots & \frac{1}{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2(n+1)} & \frac{1}{2(n+1)} & \cdots & \frac{1}{n+1} \end{pmatrix}$$

The unique solution to the linear system is $x=A^{-1}b$. With the definitions of A^{-1} and b given above, we derive the following expression for the scale degree a_k:

$$\begin{aligned} a_k &= -\frac{1}{2(n+1)}\left(\sum_{j=1}^{k-1}\omega(n-2j)\right) - \frac{\omega(n-2k)}{n+1} - \frac{1}{2(n+1)}\left(\sum_{j=k+1}^{n}\omega(n-2j)\right) \\ &= -\frac{1}{2(n+1)}\left(-2\omega n + 2\omega k\right) - \frac{\omega(n-2k)}{n+1} \\ &= \frac{\omega k}{n+1} \end{aligned}$$

This result proves that if all the weights are equal, the optimal a_k are equal subdivisions of the octave.

Appendix B

This appendix proves that equal temperament is optimal with respect to the Rasch tonality measure for any interval I_k. The diagonals of matrix M contain all instances of the interval I_k. Let $m_{k,i}$ denote the terms along the diagonal for interval I_k. The sum of these elements is:

$$\begin{aligned}\sum_{i=1}^{n+1} m_{k,i} &= a_k + \sum_{i=1}^{n-k}(a_{k+i} - a_i) + (\omega - a_{n-k+1}) + \sum_{i=n-k+2}^{n}(\omega + a_{i-(n-k+1)} - a_i)\\ &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i + \sum_{i=n-k+1}^{n}\omega\\ &= k\omega\end{aligned}$$

Therefore, the average of the instances of interval k for any scale is equal to the equal-tempered interval $k\omega/(n+1)$.

The Rasch tonality measure averages the absolute value of the differences to an ideal interval:

$$R(M,I_k) = \frac{1}{n+1}\sum_{i=1}^{n+1}\left|m_{k,i} - I_k\right|$$

Using the triangle inequality:

$$\begin{aligned}R(M,I_k) &\geq \frac{1}{n+1}\left|\sum_{i=1}^{n+1}(m_{k,i} - I_k)\right|\\ &= \frac{1}{n+1}\left|\sum_{i=1}^{n+1} m_{k,i} - \sum_{i=1}^{n+1} I_k\right|\\ &= \frac{1}{n+1}\left|k\omega - (n+1)I_k\right|\\ &= \left|\frac{k\omega}{n+1} - I_k\right|\end{aligned}$$

Therefore, for any scale, the Rasch tonality measure between the intervals k and an ideal interval I_k is always greater than the difference between the ideal interval I_k and the equal-tempered interval $k\omega/(n+1)$. In other words, for any interval, equal-temperament is optimal with respect to the Rasch tonality measure, though other tunings may also achieve the same minimum.

NOTES

1. For example, Jody Diamond (personal communication) points out that the RRI (radio station) gamelan in Surakarta, Java, is widely imitated because of the many broadcasts and recordings made on this set of instruments.

2. The theorist Erv Wilson refers to this as the "interval of equivalence" (Burt, 40).

3. For other examples of criteria sets for scales, tunings, and "musical pitch systems," see Erlich; Krumhansl (1987); Burns and Ward (264); Shepard (1987a); Dowling and Harwood (90–100). The last set of authors propose a set of "cognitive constraints on scale construction that seem to operate through much of the world" which differs from our framework in its focus on *scale*, rather than *tuning system* construction. As such, it takes into account factors—like Balzano *coherence* (or Rothenberg *propriety*), the number of intervals, and the idea of a small, modular interval—less pertinent to what Dowling and Harwood call "tonal materials" (our "tuning system"—their use of the term "tuning system" has a distinct meaning). However, our idea of key and interval weights is reflected in their "structural hierarchy" of a "modal scale."

   Yasser and others (e.g., Kraehenbuehl and Schmidt) have postulated developmental mechanisms for tuning system development. There are, of course, many different and interesting examples of composers and theorists creating their own *best* tunings, all of which in a sense, attempt something like what we suggest here: optimization with (what are often artistic) constraints. Partch's well-known *monophony* is one such example, as are the approaches of Blackwood, Lindley and Turner-Smith, and the recent work of Gräf (following Barlow) in his method of *scale rationalization* (see also Carlos).

4. The idea of a "scale" or "musical system" is considered by some authors to be a universal of human music. For example, Arom (28) says that "each society selects from the sound continuum a set of contrastive pitches. These pitches form a system, a musical scale . . . itself an abstract model but also the basis for the elaboration of all melodies, is the analog for what in a language would be its phonological system."

5. Regardless, many contemporary composers and theorists, such as Partch, Johnston, Sims, Tenney (e.g., 1987) and Fokker have devel-

oped tuning systems for their own use, with varying numbers of pitches per octave (see Keislar for an excellent survey). A number of contemporary composers working in experimental intonation have used tuning systems with an indefinite, dynamic or adaptive number of pitches per octave (for example, see Polansky 1987; 1987b). Lou Harrison referred to this approach as "free style" (Harrison 1971, 6; Polansky, 1987b). All of these strategies, while important for individual composers or smaller, often experimental musical cultures, are less effective when the tuning system needs to serve as some kind of standard for a musical community.

6. Burt surveys systems that repeat at values other than the octave, and describes some associated compositional experiments.

7. In fact, with a very simple alteration, the mathematics described later in this paper can eschew the assumption of a repeat factor.

8. Alternative just diatonic scales might use 10/9 for the major second (which occurs between the second and third and fifth and sixth scale degrees), or 27/16 for the major sixth. While these alterations shift the "wolf fifth," they have no effect on the central problem addressed here.

9. Isacoff calls it "the search for *la*," (58) and constructs the following metaphor for the problem: "Attempting to force [pure thirds, perfect octaves, and perfect fifths] into the same harpsichord or organ would be like trying to squeeze various pieces of fine furniture into a room too small to accommodate them; no matter where you place the exquisite couch, there is simply no room for the elegant loveseat. . . . The keyboard thus became a battleground of warring proportions; and the desire to achieve a tuning based on all of the ideal, simple ratios an unrealizable fantasy" (67). He writes that musical temperament is, in "application . . . similar to the Japanese art of bonsai" (95).

10. A *comma* is, in terms of our framework, a more general idea, and roughly means "the error of an actual interval to an ideal interval." Kennan suggests an algorithm for "optimally distributing" commas to "minimize the maximum of the absolute values of the errors of all the intervals that we care about."

11. Personal communication, Polansky and James Tenney. Note that Barbour (1948, 550) seems to mean something very similar when he refers to "the tuning problem." Doty (36–38) more specifically refers to the "supertonic problem": "The problems posed by the

supertonic minor triad and the syntonic comma have provoked some rather extreme reactions from composers and theorists." (Doty, 38). The ideas in this paper are one such "reaction" to the generalization of these problems.

12. "Thus there will never be a scale in which all the fifths, or a complete set of fifths and thirds, are correct. The same type of analysis shows that any method of constructing a twelve-tone scale by rational numbers is doomed to inconsistency." (Dunne and McConnell, 109). Several authors, including the latter, have employed the mathematical technique of *continued fractions* in this regard; see Berger; Brun; Barbour 1948; Rosser. Dunne and McConnell (114) even develop a 41-tone to the octave tuning using this approach.

13. Jorgensen (779) defines "wolf" quite generally, as "an interval that is considered too far out-of-tune to use in musical performance . . . its ratio is complicated."

14. There are at least three common strategies for this compromise: *equal temperament*, *well-temperament*, and what is often called *extended just intonation*. Each approaches the approximation of rational tuning spaces in a slightly different fashion. For example, equal-temperaments often search for the *minimal* even divisions of the octave which *maximally* approximate some desired ratio (such as the perfect fifth in 12-ET, or the septimal minor seventh in 31-ET). Tenney's (e.g., 1987) use of 72-ET to approximate a large number of rational intervals, as well as to explicitly use specific 72-ET degrees to "stand in" for several such intervals is another, less common version of this idea (also see Sims). Well-temperaments do something similar, but, for a variety of reasons, relax the requirement of equal system-step size. Extended just intonations, like those of Johnston, Partch, Tenney, and many others, create intonational diversity by having many intervals in the system. Partch's 43-tone monophony, for example, eventually generates *canidae* intervals of its own because of its small, finite number of ratios and primes (a fact which Partch incorporated into his work to compositional advantage). While non-tempered systems have greater accuracy within specific "keys," they require, in general, a larger number of intervals.

15. For brevity, we only show the half-matrix since the other half is the inversion of those intervals around the octave. For the complete matrix, and detailed analysis of the "supertonic problem," see Doty (36–38).

16. There are any number of definitions, in music theory, cognition, and history for terms like *scale*, *tuning*, *mode*, *tuning system*, *gamut*. Different authors mean different things and use these words more or less interchangeably. A discussion of the usages of these terms is beyond the scope of this paper, but in this example we refer to the Just Diatonic as a *scale* because it is a subset of a larger set of intervals. Some of these, like the half-step 16/15, and the minor third 6/5, do not explicitly appear in this representation. They would if the scale were written as adjacent (rather than absolute) intervals: 1/1 9/8 10/9 16/15 9/8 10/9 9/8 16/15. The intervals do appear in the half-matrix, but share diagonals with other intervals of different categories: both the major second (9/8) and the minor second (16/15) appear in the same diagonals. The interval matrix of the larger tuning system from which the Just Diatonic scale is derived, would more clearly distinguish intervals.

17. Rasch says that "Few if any historical tunings have received more attention in the literature than 'Werckmeister III'" (29). Donahue says of W3: "It may be the first documented temperament for keyboard instruments that did not have a wolf interval" (26). Similarly, our framework is intended to produce tunings which intentionally minimize "canidae intervals."

18. In these preliminary examples, it suffices to show the half-matrix, although in our mathematical framework, for computational reasons, the error function is taken on the complete matrix. The interval matrix is symmetric around the repeat factor. Values below the diagonal are intervals that span the repeat factor until the next occurrence (in this case, two octaves of the tuning), so that each "key" is represented by the same number of intervals.

19. The L^1 and L^2 measures are but two in a continuous range of L^p measures, defined by taking the p^{th} root of the sum of the p^{th} powers (for any positive real number p) of the absolute value of the differences. Both the L^1 and L^2 functions measure pitch distance, but other error functions, such as those which measure the harmonic distance of the intervals (like the Euler *gradus suavitatis* (GS) function, the Tenney *harmonic distance* function (Tenney, 1988; see Chalmers 1983 for a comparison of these two functions), or the Barlow *harmonicity* function (Barlow)) are possible as well. These functions suggest interesting ways to extend the framework in this paper (see, for instance Gräf). John Rahn (personal communication) has suggested the possibility of incorporating *sign* in the error function, rather than always using absolute value, an extension that

might have interesting ramifications for the way the framework represents inversion.

20. In all cases tested, the genetic-algorithm solution with the L^1 measure finds the same solution as the deterministic one with the L^2 measure (see Software). However this non-deterministic approach offers the important possibility of easily substituting alternative error functions, and experimenting with other aspects of the framework, including perhaps, what Rahn (personal communication) refers to as the "structure of the error" itself.

21. In the weight matrix W, the key and interval weights are multiplied. As a result, a weight matrix cannot be uniquely decomposed into its component matrices—the individual matrices can be scaled up and down by the same factor without affecting the product. If, however, the sets of key and interval weights are constrained to have unit L^2 norm, a weight matrix can be uniquely decomposed into key and interval weights.

22. This is due in large part to the particular construction of our framework, which incorporates redundancy. Each cell in the matrix is weighted by the product of the key and interval weights, allowing for (we hope) a more general and musically intuitive specification of tuning system features. However, it is a simple matter to alter the way these weights are specified. For example, a single *weight matrix* might be used, specifying a specific value for each cell. In this way, a greater degree of weighting specificity can be achieved, although, in some cases, at the expense of creating a less useful model.

23. Note that key weights are shown for each "step" of the tuning system, which correspond to the top "row" of the interval matrix. In other words, in this example the first "key" of the scale, which might arbitrarily be called "that key starting on C," has the highest weight. Similarly, the "key starting on G" has the next highest key weight.

24. Tuning systems are often generated from a small number of important intervals, like the fifth and third. Using heuristic search algorithms and our framework, we have been able to generate, from small sets of ideal intervals (weighting all others to zero) a number of documented historical and world music scales. Mathematically, this is not surprising, but this technique suggests possibilities for future work in the area.

25. For example keys beginning on the first, fourth, and fifth notes (*tonic*, *subdominant,* and *dominant*) are thought to be "central" or inner in tonal music, an assumption usually reflected in the tuning systems themselves. Similar hierarchies exist in other musical cultures, such as in Central Javanese music.

26. Chalmers suggests a least-squares fit for the minimization of specific interval error, an idea influential to this current framework. Sethares (1997, 69–70), introducing his own tuning optimization algorithm (also Sethares 1993) based on the best-fit to particular spectra, surveys a number of tuning evaluation methods (aptly titled "My Tuning is Better Than Yours").

27. Rasch's function is slightly different than the measure of thirds, fourths, and fifths used in Table 4.

28. Given some fixed number of pitches and of ideal intervals, it can be proved that ET achieves the minimum mean-tempering value, but so do other tuning systems (see Appendix B).

29. Given a set of ideal intervals, our framework can generate infinitely many scales, including preexisting ones (like W3). For this reason it is worth clarifying the methodology of these examples.

    We specified a small subset of ideal intervals (OWT1 = [5/4 4/3 3/2 8/5]; OWT2 = [9/8 7/6 5/4 4/3 3/2 8/5 12/7 16/9]) and searched for an accompanying *set of weights* (key and interval) which generated an optimal scale with the specified minimal *mean-tempering value*. To perform this search, we used a standard *hill-climbing algorithm* (in the program Matlab).

    The weights found by the search algorithm were:

OWT1

interval		1	1	1	150	1600	1	1	50	1	1	1
key	50	60	7	1	7	60	50	60	7	1	7	80

OWT2

interval		1	1	1	10	2000	1	2000	10	1	1	1
key	50	30	10	30	1	30	10	30	50	1	10	1

30. Several theorists and composers have discussed the importance and prevalence of septimal tunings. For example see Erlich; Fokker; Harrison; Doty; Partch. "So it seems natural to try and expand harmony by making 7-limit intervals as fundamental harmonic units.

The new consonant intervals would be 7:4, 7:5 and 7:6." (Erlich, 2). Doty (47–54) discusses some historical and "ethnic" seven-limit tunings. "Thus, while it may not seem accurate to say that no dominant seventh chords or diminished triads exist in the five-limit, the seven-limit versions of these chords offer such superior consonance and clarity that they are to be preferred in almost every case . . ." (Doty, 44). Fokker's widely used 31-ET tuning has, in fact a more accurate approximation of what Rasch (Fokker, 27) calls the "harmonic seventh" than the 3/2 perfect fifth. Polansky (1984), citing Harrison and Surjodiningrat et al., discusses septimal intervals in slendro gamelan tunings.

31. Generally, we do not use 0 for weights, but some very small positive number (like 0.001). For mathematical reasons whose description is outside the scope of this paper, zero weights cause the system to be underconstrained.

32. The five pitches plus the octave are numbered 1, 2, 3, 5, 6, 1'. There is no 4 in slendro (4 and 7 appear in pelog).

33. It is not completely clear that this last note is a 5 in Sumarsam's redrawing of Sindusawarno's chart, but it appears likely.

34. In pathet nem, which is sometimes said to be a kind of "mixture" of manyura and sanga, the kempyung relationships are to 5, not adjacent. See also Perlman and Krumhansl: "Only two Javanese interval labels are commonly known, and they seem to refer primarily to instrumental technique, not tonal distance" (99).

35. "The apparently clear-cut, formal simplicity of the logic of the pathet system belies its actual complexity . . . pathet has stimulated and frustrated generations of Javanese theorists and has become one of the most heavily researched topics in gamelan scholarship. Making sense of it is a task that must await another occasion." (Perlman, 42–43).

36. Perlman states that in his own study of "22 fixed-pitch instruments in the slendro tuning system, no interval was found to vary less than 30 cents, and some varied as much as 75 cents." In the GM study (Table 8), the minimum intervallic variation is 32 cents, the maximum 52 cents.

37. We are only using Table 8 of the GM study: "Classification of 28 outstanding slendro gamelans in Jogyakarta and Surakarta in accordance with their laras based on the pitches of saron demung or gender barung." Measuring the central octave of the gendér barung

is common as a basis for comparing gamelan tunings. Additionally, we only use twenty-seven of the twenty-eight gamelans, since there seems to be a 10 cent error in gamelan #25 (Gamelan Pengawesari, from the Pakualaman Jogyakarta) in the GM study: it should add up to 1213 cents (oddly, like gamelan #24), not 1223 cents.

38. It makes sense to fix either key or interval weights: if both are left indeterminate, their interdependence makes experimental results more difficult to interpret. In other words, in searching for either one or the other, the search space is significantly smaller.

39. Our use of the term fifth is a bit misleading mathematically: the interval is in fact a spread of four scale degrees. Its Javanese name is *kempyung*, but as it spans more or less the pitch distance of a perfect fifth in western music, we use it here to avoid confusion.

40. We used this minimal set (2) of rational intervals for their acoustic and cognitive simplicity. For the second, 9/8 or 10/9 would be arguably as simple, numerically and acoustically. However both are much narrower than any interval in the GM data.

    Using only two ideal intervals also parallels our resynthesis method for historical WTs above. These intervals (but not necessarily these tunings) also seem to be important to musical style (cadences on the fifth, frequent movement by seconds) and the tunings are somewhat motivated by the statistical data. We hope that further studies will experiment with a number of other intervallic possibilities (a likely candidate is the interval mentioned above, 267 cents).

41. These numbers are larger than the average errors in *Experiment 1* because our constraints are stronger: five intervals rather than two. For some reason, as well, there seems to be a subtle mathematical difference between fixing key weights and fixing interval weights which results in a more difficult search in the former case. This difference is not yet well understood.

42. The framework might be extended to facilitate the choice of several weighted alternatives for certain ideal intervals, such as the aforementioned second in the Just Diatonic scale. For example, the error function might use the *minimum* error to the 81/64 or 5/4, rather than the *actual* error to one of them.

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Software: MatLab and Java tools for this framework, including a real-time MIDI application for exploring optimal tunings (written by Wei Pan) are available at http://eamusic.dartmouth.edu/~larry/owt/index.html

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