4 The construction of new genera

THIS CHAPTER IS concerned with the construction of new genera in addition to those collated from the texts of the numerous classical, medieval, and recent writers. The new tetrachords are a very heterogeneous group, since they were generated by the author over a period of years using a number of different processes as new methods were learned or discovered. Including historical tetrachords, the tabulated genera in the catalogs number 723, of which 476 belong in the Main Catalog, 16 in the reduplicated section, 101 under miscellaneous, 98 in the tempered list, and 32 in the semi-tempered category.

The genera in the Main Catalog are classified according to the size of their largest or *characteristic interval* (CI) in decreasing order from 13/10 (454 cents) to 10/9 (182 cents). There are 73 CIs acquired from diverse historical and theoretical sources (4-1). Sources are documented in the catalogs. The theoretical procedures for obtaining the new genera are described in this chapter and the next.

New genera derived by linear division

The first of the new genera are those whose CIs are relatively simple non-superparticular ratios such as 11/9, 14/11, and 16/13. These ratios were drawn initially from sources such as Harry Partch's 43-tone, 11-limit just intonation gamut, but it was discovered later that some of these CIs are to be found in historical sources as well. The second group is composed of intervals such as 37/30, which were used sporadically by historical writers. To these ratios may be added their 4/3's and 3/2's complements, e.g. 27/22

4-1. *Characteristic intervals (CIs) of new genera in just intonation. The CI is the largest interval of the tetrachord and the pyknon or apyknon is the difference between the CI and the fourth. Because many of the new genera have historically known CIs, all of the CIs in the Main Catalog are listed in this table. The CIs of the reduplicated, miscellaneous, tempered, and semi-tempered lists are not included in this table.*

HYPERENHARMONIC GENERA

The term hyperenharmonic is originally from Wilson and refers to genera whose CI is greater than 425 cents. The prototypical hyperenharmonic genus is Wilson's 56/55 · 55/54 · 9/7. See chapter 5 for classification schemes.

	CI	PYKNON	CENTS
H1	13/10	40/39	454 + 44
H2	35/27	36/35	449 + 49
H3	22/17	34/33	446 + 52
H4	128/99	33/32	445 + 53
H5	31/24	32/31	443 + 55
H6	40/31	31/30	441 + 57
H7	58/45	30/29	439 + 59
H8	9/7	28/27	435 + 63
H9	104/81	27/26	433 + 65
H10	50/39	26/25	430 + 68
H11	32/25	25/24	427 + 71

ENHARMONIC GENERA

The CIs of the enharmonic genera range from 375 to 425 cents.

E1	23/18	24/23	424 + 73
E2	88/69	23/22	421 + 77
E3	50/41	160/153	421 + 77
E4	14/11	22/21	418 + 81
E5	80/63	21/20	414 + 84
E6	33/26	104/99	413 + 85
E7	19/15	20/19	409 + 89
E8	81/64	256/243	408 + 90
E9	24/19	19/18	404 + 94
E10	34/27	18/17	399 + 99
E11	113/90	120/113	394 + 104
E12	64/51	17/16	393 + 105
E13	5/4	16/15	386 + 112
E14	8192/6561	2187/2048	384 + 114
E15	56/45	15/14	379 + 119
E16	41/33	44/41	376 + 122

CHROMATIC GENERA

The CIs of the chromatic genera range from 375 to 250 cents.

C1	36/29	29/27	374 + 124
C2	26/21	14/13	370 + 128
C3	21/17	68/63	366 + 132
C4	100/81	27/25	365 + 133
C5	37/30	40/37	363 + 135
C6	16/13	13/12	359 + 139
C7	27/22	88/81	355 + 143
C8	11/9	12/11	347 + 151
C9	39/32	128/117	342 + 156
C10	28/23	23/21	341 + 157
C11	17/14	56/51	336 + 162
C12	40/33	11/10	333 + 165
C13	29/24	32/29	328 + 170
C14	6/5	10/9	316 + 182
C15	25/21	28/25	302 + 196
C16	19/16	64/57	298 + 201
C17	32/27	9/8	294 + 204
C18	45/38	152/135	293 + 205
C19	13/11	44/39	289 + 209
C20	33/28	112/99	284 + 214
C21	20/17	17/15	281 + 217
C22	27/23	92/81	278 + 220
C23	75/64	256/225	275 + 223
C24	7/6	8/7	267 + 231
C25	136/117	39/34	261 + 238
C26	36/31	31/27	259 + 239
C27	80/69	23/20	256 + 242
C28	22/19	38/33	254 + 244
C29	52/45	15/13	250 + 248

DIATONIC GENERA

The CIs of the diatonic genera range from 250 to 166 cents. In the diatonic genera, a pyknon does not exist.

D1	15/13	52/45	248 + 250
D2	38/23	22/19	242 + 256
D3	23/20	80/69	242 + 256
D4	31/27	36/31	239 + 259
D5	39/34	136/117	238 + 261
D6	8/7	7/6	231 + 267
D7	256/225	75/64	223 + 275
D8	25/22	88/75	221 + 277
D9	92/81	27/23	220 + 278
D10	76/67	67/57	218 + 280
D11	17/15	20/17	217 + 281
D12	112/99	33/28	214 + 284
D13	44/39	13/11	209 + 289
D14	152/135	45/38	205 + 293
D15	9/8	32/27	204 + 294
D16	160/143	143/120	194 + 304
D17	10/9	6/5	182 + 316

4-2. *Indexed genera. The terms 4 and 3 which represent the 1/1 and 4/3 of the final tetrachord are multiplied by the index. The lefthand sets of tetrachords are those generated by selecting and recombining the successive intervals resulting from the additional terms after the multiplication. The righthand sets of tetrachords have been reduced to lowest terms and ordered with the CI uppermost.*

MULTIPLIER: 4 TERMS: 16 15 14 13 12

16/15 · 15/14 · 14/12	16/15 · 15/14 · 7/6
16/15 · 15/13 · 13/12	16/15 · 13/12 · 15/13
16/14 · 14/13 · 13/12	14/13 · 13/12 · 8/7

MULTIPLIER: 5 TERMS: 20 19 18 17 16 15

20/19 · 19/18 · 18/15	20/19 · 19/18 · 6/5
20/19 · 19/17 · 17/15	20/19 · 19/17 · 17/15
20/19 · 19/16 · 16/15	20/19 · 16/15 · 19/16
20/18 · 18/17 · 17/15	18/17 · 10/9 · 17/15
20/18 · 18/16 · 16/15	16/15 · 10/9 · 9/8
20/17 · 17/16 · 16/15	17/16 · 16/15 · 20/17

MULTIPLIER: 6 TERMS: 24 23 22 21 20 19 18

24/23 · 23/22 · 22/18	24/23 · 23/22 · 11/9
24/23 · 23/21 · 21/18	24/23 · 23/21 · 7/6
24/23 · 23/20 · 20/18	24/23 · 10/9 · 23/20
24/23 · 23/19 · 19/18*	24/23 · 19/18 · 23/19
24/22 · 22/21 · 21/18	22/21 · 12/11 · 7/6
24/22 · 22/20 · 20/18	12/11 · 11/10 · 10/9
24/22 · 22/19 · 19/18	19/18 · 12/11 · 22/19
24/21 · 21/20 · 20/18	21/20 · 10/9 · 8/7
24/21 · 21/19 · 19/18	19/18 · 21/19 · 8/7
24/20 · 20/19 · 19/18	20/19 · 19/18 · 6/5

* *see Catalog number 536.*

is the 3/2's complement of 11/9 and 52/45 the 4/3's complement of 15/13. Various genera were then constructed by dividing the pykna or apykna by linear division into two or three parts to produce 1:1, 1:2, and 2:1 divisions. Both the 1:2 and 2:1 divisions were made to locate genera composed mainly of superparticular ratios. Even Ptolemy occasionally had to reorder the intervals resulting from triple division before recombining two of them to produce the two intervals of the pyknon (2-2 and 2-4). More complex divisions were found either by inspection or by katapyknosis with larger multipliers.

Indexed genera

One useful technique, originated by Ervin Wilson, is a variation of the katapyknotic process. In 4-2 this technique is applied to the 4/3 rather than to the pyknon (as it was in 2-4). The 1/1 and 4/3 of the undivided tetrachord are expressed as 3 and 4, and are multiplied by a succession of numbers of increasing magnitude. The new terms resulting from such a multiplication and all the intermediate numbers define a set of successive intervals which may be sequentially recombined to yield the three intervals of tetrachords. I have termed the multiplier, the index, and the resulting genera *indexed genera*. The intermediate terms are a sequence of arithmetic means between the extremes.

The major shortcoming of this procedure is that the number of genera grows rapidly with the index. There are 120 genera of index 17, and not all of these are worth cataloguing, since other genera of similar melodic contours and simpler ratios are already known and tabulated. The technique is still of interest, however, to generate sets of tetrachords with common numerical relations for algorithmic composition.

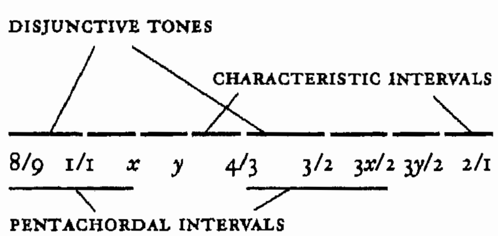
Pentachordal families

Archytas's genera were devised so that they made the interval 7/6 between their common first interval, 28/27, and the note a 9/8 below the first note of the tetrachord (Erickson 1965; Winnington-Ingram 1932; see also 6-1). Other first intervals *(x)* may be chosen so that in combination with the 9/8 they generate harmonically and melodically interesting intervals. These intervals may be termed *pentachordal intervals (PI)* as they are part of a pentachordal, rather than a tetrachordal tonal sequence. Three such groups or families of tetrachords are given in 4-3 along with their initial and pentachordal intervals.

The 28/27 family is an expansion of Archytas's set of genera. The 40/39 family fits quite well into 24-tone equal temperament because of the reasonably close approximation of many of the ratios of 13 to quarter-tone intervals. The 15/13 is another plausible tuning for the interval of five dieses which was reputed to be a feature of the oldest scales (chapter 6; Bacchios, 320 CE in Steinmayer 1985). The 16/15 family contains the most consonant tunings of the chromatic and diatonic genera.

The pentachordal intervals of 4-3 are the *mediants* ("thirds") of the triads which generate the *tritriadic* scales of chapter 7, where they are discussed in greater detail. In general, all tetrachords containing a medial 9/8 may function as generators of tritriadic scales.

4-3. *Pentachordal intervals and families. These tetrachords are defined by two parameters: the pentachordal interval, 9x/8, and the characteristic interval, which determines the genus. An initial interval* x *results in a pentachordal interval (PI) of 9x/8. These pentachordal families are the most important tritriadic genera of chapter 7. The initials are the first intervals of the tetrachords.*



INITIAL	PI	INITIAL	PI	INITIAL	PI
16/15	6/5	10/9	5/4	8/7	9/7
28/27	7/6	12/11	27/22	88/81	11/9
13/12	39/32	128/117	16/13	22/21	33/28
112/99	14/11	40/39	15/13	52/45	13/10
44/39	33/26	104/99	13/11	56/51	21/17
68/63	17/14	64/57	24/19	19/18	19/16
256/243	32/27	9/8	81/64	52/51	39/34
136/117	17/13	7/6	21/16	64/63	8/7
80/68	30/23	56/45	23/20	24/23	27/23
92/81	23/18	184/171	57/46	76/69	23/19

x = 40/39, PI = 15/13

ENHARMONIC	
40/39 · 39/38 · 19/15	ERATOSTHENES
40/39 · 26/25 · 5/4	AVICENNA
CHROMATIC	
40/39 · 13/12 · 6/5	BARBOUR
40/39 · 39/35 · 7/6	
40/39 · 11/10 · 13/11	
DIATONIC	
40/39 · 52/45 · 9/8	
40/39 · 91/80 · 8/7	

x = 28/27, PI = 7/6

ENHARMONIC	
28/27 · 36/35 · 5/4	ARCHYTAS
CHROMATIC	
28/27 · 243/224 · 32/27	ARCHYTAS
28/27 · 15/14 · 6/5	PTOLEMY
28/27 · 27/26 · 26/21	MAIN CATALOG
DIATONIC	
28/27 · 8/7 · 9/8	ARCHYTAS
28/27 · 39/35 · 15/13	MAIN CATALOG

x = 16/15, PI = 6/5

CHROMATIC	
16/15 · 25/24 · 6/5	DIDYMOS
16/15 · 15/14 · 7/6	AL-FARABI
16/15 · 20/19 · 19/16	KORNERUP
DIATONIC	
16/15 · 9/8 · 10/9	PTOLEMY
16/15 · 13/12 · 15/13	MAIN CATALOG

Mean tetrachords

The mathematician and musician Archytas may have been the first to recognize the importance of the arithmetic, harmonic, and geometric means to music. He was credited with renaming the mean formerly called the "subcontrary" as the harmonic mean because it produced more pleasing melodic divisions than the arithmetic mean (Heath [1921] 1981; Erickson 1965). His own tunings were constructed by the application of only the harmonic and arithmetic means, but there were actually nine other means known to Greek mathematicians and which might be used to construct tetrachords (Heath [1921] 1981).

To this set of twelve may be added the *root mean square* or *quadratic mean* and four of my own invention whose definitions are given along with the historical ones in 4-4. The logarithmic mean divides an interval into two parts, the ratio of whose widths is the inverse of the ratio of the extremes of the interval. For example, the logarithmic mean divides the 2/1 into two

4-4. *Means: formulae and equivalent expressions from Heath 1921, 1:85–87, except for the logarithmic, ratio, and root mean square means. Number 12 is the framework of the scale when* a = 12 *and* b = 6. *The tetrachords generated by number 17 are extremely close numerically to the counter-logarithmic mean tetrachords of the other kinds. They also resemble the subcontraries to the geometric means.*

1. ARITHMETIC
$(a-b)/(b-c) = a/a = b/b = c/c \quad a+c = 2b$

2. GEOMETRIC
$(a-b)/(b-c) = a/b = b/c \quad ac = b^2$

3. HARMONIC
$(a-b)/(b-c) = a/c, 1/a + 1/c = 2/b \quad b = 2ac/(a+c)$

4. SUBCONTRARY TO HARMONIC
$(a-c)/(b-c) = c/a \quad (a^2+c^2)/(a+c) = b$

5. FIRST SUBCONTRARY TO GEOMETRIC
$(a-b)/(b-c) = c/b \quad a = b+c-c^2/b$

6. SECOND SUBCONTRARY TO GEOMETRIC
$(a-b)/(b-c) = b/a \quad c = a+b-a^2/b$

7. UNNAMED
$(a-c)/(b-c) = a/c \quad c^2 = 2ac - ab$

8. UNNAMED
$(a-c)/(a-b) = a/c \quad a^2+c^2 = a(b+c)$

9. UNNAMED
$(a-c)/(b-c) = b/c \quad b^2+c^2 = c(a+b)$

10. UNNAMED (SAME AS FIBONACCI SERIES)
$(a-c)/(a-b) = b/c \quad a = b+c$

11. UNNAMED
$(a-c)/(a-b) = a/b \quad a^2 = 2ab - bc$

12. MUSICAL PROPORTION
$a : (a+b)/2 = 2ab/(a+b) : b$

13. LOGARITHMIC MEAN
$\log b = (c \log a + a \log c)/(a+c) \quad (b/a)^c = (c/b)^a$

14. COUNTER-LOGARITHMIC MEAN
$\log b = (a \log a + c \log c)/(a+c) \quad (b/a)a = (c/b)c$

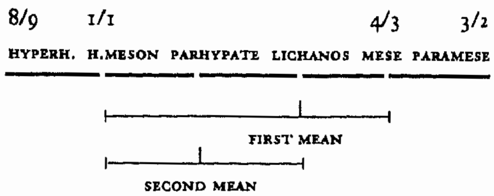
15. RATIO MEAN
$(a-c)/(b-c) = x/y \quad c = (bx-ay)/(x-y)$

16. SECOND RATIO MEAN
$(a-c)/(a-b) = x/y \quad c = (ay-ax+bx)/y$

17. ROOT MEAN SQUARE
$b = \sqrt{((a^2+c^2)/2)} \quad b^2 = (a^2+c^2)/2$

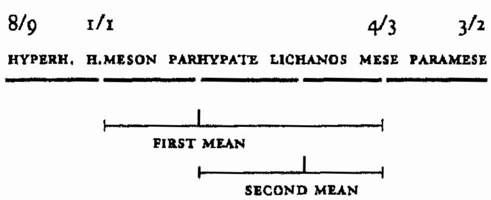
4-5. *Generating tetrachords with means.*

MEAN TETRACHORDS OF THE FIRST KIND



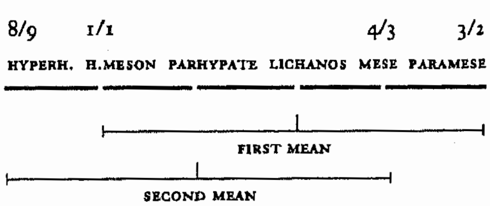
Lichanos is defined as the appropriate mean between hypate meson (1/1) and mese (4/3). Parhypate is then computed as the identical mean between lichanos and hypate.

MEAN TETRACHORDS OF THE SECOND KIND



Parhypate is defined as the appropriate mean between hypate meson (1/1) and mese (4/3). Lichanos is then computed as the identical mean between parhypate and mese.

MEAN TETRACHORDS OF THE THIRD KIND



Lichanos is defined as the appropriate mean between hypate meson (1/1) and paramese (3/2). Parhypate is then computed as the identical mean between mese (4/3) and hyperhypate (8/9).

intervals of 400 and 800 cents in the proportion of 1:2 (0, 400, and 1200 cents). The *counter-logarithmic mean* effects the same division in the opposite order, i.e., 800 and 400 cents (0, 800, and 1200 cents).

The two *ratio means*, numbers 15 and 16, are variations of numbers 7 and 8 of 4-4, differing only in that the ratio of the difference of the extremes to the difference between the mean and one of the extremes is dependent upon the parameter x/y.

There are still other types of mean, but these seventeen are sufficient to generate a considerable number of tetrachords (4-6–8) and may be of further utility in the algorithmic generation of melodies.

The most obvious procedures for generating tetrachords from these means are shown in 4-5. Mean tetrachords of the first kind are constructed by first calculating the lichanos as the mean between 1/1 and 4/3, or equivalently between $a = 4$ and and $c = 3$. The next step is the computation of parhypate as the same mean between 1/1 and the just calculated lichanos (4-6). Tetrachords of the second kind have the mean operations performed in reverse order (4-7). Tetrachords of the third kind are found by taking the means between 1/1 and 3/2 and between 8/9 and 4/3 (4-8); the smaller is defined as parhypate; the larger becomes the lichanos.

The construction of sets of genera analogous to those of Archytas, which are composed of a mean between 8/9 and 4/3 and its "subcontrary" or "counter"-mean between 8/9 and 32/27 (Erickson 1965; Winnington-Ingram 1932), is left for future investigations as it involves deep questions about the integration of intervals into musical systems.

Multiple means may be defined for the arithmetic, harmonic, and geometric means. The insertion of two arithmetic or harmonic means into the 4/3 results in Ptolemy's equable diatonic and its intervallic retrograde, 12/11 · 11/10 · 10/9, 10/9 · 11/10 · 12/11. The geometric mean equivalent is the new genus 166.667 + 166.667 + 166.667 cents (see the discussion of tempered tetrachords below).

4-6. *Mean tetrachords of the first kind. The lichanoi are the means between 1/1 and 4/3; the parhypatai are the means between 1/1 and the lichanoi.*

1. ARITHMETIC	1/1 13/12 7/6 4/3	13/12 · 14/13 · 8/7	139 + 128 + 231
2. GEOMETRIC	1.0 1.07457 1.15470 1.33333	1.07457 · 1.07457 · 1.15470	125 + 125 + 249
3. HARMONIC	1/1 16/15 8/7 4/3	16/15 · 15/14 · 7/6	112 + 119 + 267
4. SUBCONTRARY TO HARMONIC	1/1 533/483 25/21 4/3	533/483 · 575/533 · 28/25	171 + 131 + 196
5. FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.09429 1.18046 1.33333	1.09429 · 1.07874 · 1.12950	156 + 131 + 211
6. SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.09185 1.17704 1.33333	1.09185 · 1.07803 · 1.13278	152 + 130 + 216
7. UNNAMED	1/1 6/5 5/4 4/3	6/5 · 25/24 · 16/15	316 + 71 + 112
8. UNNAMED	1/1 157/156 13/12 4/3	157/156 · 169/157 · 16/13	11 + 128 + 359
9. UNNAMED	1.0 1.21677 1.26376 1.33333	1.21677 · 1.03862 · 1.05505	340 + 66 + 93
10. FIBONACCI SERIES	NO SOLUTION	—	—
11. UNNAMED	1/1 256/255 16/15 4/3	256/255 · 17/16 · 5/4	7 + 105 + 386
12. MUSICAL PROPORTION	1/1 8/7 7/6 4/3	8/7 · 49/48 · 8/7	231 + 36 + 231
13. LOGARITHMIC MEAN	1.0 1.05956 1.13122 1.33333	1.05956 · 1.06763 · 1.17867	100 + 113 + 285
14. COUNTER-LOGARITHMIC MEAN	1.0 1.09301 1.17867 1.33333	1.09301 · 1.07837 · 1.13122	154 + 131 + 213
15. RATIO MEAN (X/Y = 4/3)	1/1 19/16 5/4 4/3	19/16 · 20/19 · 16/15	298 + 89 + 112
16. SECOND RATIO MEAN (X/Y = 4/3)	1/1 157/156 13/12 4/3	157/156 · 169/157 · 16/13	11 + 128 + 359
17. ROOT MEAN SQUARE	1.0 1.09290 1.17851 1.33333	1.09291 · 1.078328 · 1.13137	154 + 131 + 214

4-7. *Mean tetrachords of the second kind. The parhypatai are the means between 1/1 and 4/3; the lichanoi are the means between the parhypatai and 4/3.*

1. ARITHMETIC	1/1 7/6 5/4 4/3	7/6 · 15/14 · 16/15	267 + 119 + 112
2. GEOMETRIC	1.0 1.15470 1.24081 1.33333	1.15470 · 1.07457 · 1.07457	249 + 125 + 125
3. HARMONIC	1/1 8/7 16/13 4/3	8/7 · 14/13 · 13/12	231 + 128 + 139
4. SUBCONTRARY TO HARMONIC	1/1 25/21 1409/1113 4/3	25/21 · 1409/1325 · 1484/1409	302 + 106 + 90
5. FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.18046 1.25937 1.33333	1.18046 · 1.06685 · 1.05873	287 + 112 + 99
6. SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.17704 1.25748 1.33333	1.17704 · 1.06833 · 1.06032	282 + 114 + 101
7. UNNAMED	1/1 5/4 85/64 4/3	5/4 · 17/16 · 256/255	386 + 105 + 7
8. UNNAMED	1/1 13/12 217/192 4/3	13/12 · 217/208 · 256/217	139 + 73 + 286
9. UNNAMED	1.0 1.26376 1.3299 1.33333	1.26376 · 1.05321 · 1.00260	405 + 88 + 4
10. FIBONACCI SERIES	NO SOLUTION	—	—
11. UNNAMED	1/1 16/15 10/9 4/3	16/15 · 25/24 · 6/5	112 + 71 + 316
12. MUSICAL PROPORTION	1/1 8/7 7/6 4/3	8/7 · 49/48 · 8/7	231 + 36 + 23
13. LOGARITHMIC MEAN	1.0 1.13122 1.21987 1.33333	1.13122 · 1.07837 · 1.09301	213 + 131 + 154
14. COUNTER-LOGARITHMIC MEAN	1.0 1.17867 1.25839 1.33333	1.17867 · 1.06763 · 1.05956	285 + 113 + 100
15. RATIO MEAN (X/Y=4/3)	1/1 5/4 21/16 4/3	5/4 · 21/20 · 64/63	386 + 84 + 27
16. RATIO MEAN (X/Y=4/3)	1/1 13/12 55/48 4/3	13/12 · 55/52 · 64/55	139 + 97 + 262
17. ROOT MEAN SQUARE	1.0 1.17851 1.22583 1.33333	1.17851 · 1.067708 · 1.059625	284 + 113 + 100

Summation tetrachords

Closely related to these applications of the various means is a simple nique which generates certain historically known tetrachords as v some unusual divisions. Wilson has called this *freshman sums*, and h plied it in many different musical contexts (Wilson 1974, 1986, 1989 numerators and denominators of two ratios are summed separat obtain a new fraction of intermediate size (Lloyd and Boyle 1978 example, the freshman sum of 1/1 and 4/3 is 5/4 , and the sum of 5 1/1 is 6/5. These ratios define the tetrachord 1/1 6/5 5/4 4/3. Similar "sum" of 5/4 and 4/3 is 9/7, and these ratios delineate the 1/1 5/4 9 tetrachord. The former is a permutation of Didymos's chromatic gen the latter is the inversion of Archytas's enharmonic. If one emp multiplier/index as in 4-2 and expresses the 1/1 as 2/2, 3/3. . . , an i set of graded tetrachords may be generated. The most important a teresting ones are tabulated in 4-9.

Similarly, the multiplier may be applied to the 4/3 rather than the yield 8/6, 12/9. . . . The resulting tetrachords fall into the enharmor hyperenharmonic classes and very quickly comprise intervals too sr be musically useful. A few of the earlier members are listed in 4-10.

4-8. *Mean tetrachords of the third kind. The lichanoi of these tetrachords are the means between 1/1 and 3/2; the parhypatai are the means between 8/9 and 4/3. These tetrachords are also tritriadic genera.*

1.	ARITHMETIC	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
2.	GEOMETRIC	1.0 1.08866 1.22474 1.33333	1.08866 · 1.125 · 1.08866	147 + 204 + 14
3.	HARMONIC	1/1 16/15 6/5 4/3	16/15 · 9/8 · 10/9	112 + 204 + 18
4.	SUBCONTRARY TO HARMONIC	1/1 52/45 13/12 4/3	52/45 · 9/8 · 40/39	250 + 204 + 44
5.	FIRST SUBCONTRARY TO GEOMETRIC	1.0 1.13847 1.28078 1.33333	1.13847 · 1.125 · 1.0410	225 + 204 + 70
6.	SECOND SUBCONTRARY TO GEOMETRIC	1.0 1.12950 1.27069 1.33333	1.1295 · 1.125 · 1.0493	211 + 204 + 83
7.	UNNAMED	NO SOLUTION	—	—
8.	UNNAMED	1/1 28/27 7/6 4/3	28/27 · 9/8 · 8/7	63 + 204 + 231
9.	UNNAMED	NO SOLUTION	—	—
10.	FIBONACCI SERIES	NO SOLUTION	—	—
11.	UNNAMED	NO SOLUTION	—	—
12.	MUSICAL PROPORTION	NOT DEFINED	—	—
13.	LOGARITHMIC MEAN	1.0 1.04540 1.17608 1.33333	1.0454 · 1.125 · 1.1337	77 + 204 + 217
14.	COUNTER-LOGARITHMIC MEAN	1.0 1.13371 1.27542 1.33333	1.1337 · 1.125 · 1.0454	217 + 204 + 77
15.	RATIO MEAN (X/Y = 2/1)	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
16.	RATIO MEAN (X/Y = 2/1)	1/1 10/9 5/4 4/3	10/9 · 9/8 · 16/15	182 + 204 + 11
17.	ROOT MEAN SQUARE	1.0 1.1331 1.27475 1.33333	1.1331 · 1.125 · 1.04595	216 + 204 + 78

4-9. *Summation tetrachords of the first type. Unreduced ratios have been retained to clarify the generating process.*

	TETRACHORD	RATIOS	SOURCE
1.	1/1 6/5 5/4 4/3	6/5 · 25/24 · 16/15	DIDYMOS
2.	1/1 5/4 9/7 4/3	5/4 · 36/35 · 28/27	ARCHYTAS
3.	2/2 8/7 6/5 4/3	8/7 · 21/20 · 10/9	PTOLEMY
4.	2/2 6/5 10/8 4/3	6/5 · 25/24 · 16/15	DIDYMOS
5.	3/3 10/9 7/6 4/3	10/9 · 21/20 · 8/7	PTOLEMY
6.	3/3 7/6 11/9 4/3	7/6 · 22/21 · 12/11	PTOLEMY
7.	4/4 12/11 8/7 4/3	12/11 · 22/21 · 7/6	PTOLEMY
8.	4/4 8/7 12/10 4/3	8/7 · 21/20 · 10/9	PTOLEMY
9.	5/5 14/13 9/8 4/3	14/13 · 117/112 · 32/27	MISC. CAT.
10.	5/5 9/8 13/11 4/3	9/8 · 104/99 · 44/39	MAIN CAT.
11.	6/6 16/15 10/9 4/3	16/15 · 25/24 · 6/5	DIDYMOS
12.	6/6 10/9 14/12 4/3	10/9 · 21/20 · 7/6	PTOLEMY
13.	7/7 18/17 11/10 4/3	18/17 · 187/180 · 40/33	MISC. CAT.
14.	7/7 11/10 15/13 4/3	11/10 · 150/143 · 52/45	MISC. CAT.
15.	8/8 20/19 12/11 4/3	20/19 · 57/55 · 11/9	MAIN CAT.
16.	8/8 12/11 16/14 4/3	12/11 · 22/21 · 7/6	PTOLEMY
17.	9/9 22/21 13/12 4/3	22/21 · 91/88 · 16/13	MISC. CAT.
18.	9/9 13/12 17/15 4/3	13/12 · 68/65 · 20/17	MAIN CAT.
19.	10/10 24/23 14/13 4/3	24/23 · 161/156 · 26/21	MISC. CAT.
20.	10/10 14/13 18/16 4/3	14/13 · 117/112 · 32/27	MISC. CAT.
21.	11/11 26/25 15/14 4/3	26/25 · 375/364 · 56/45	MISC. CAT.
22.	11/11 15/14 19/17 4/3	15/14 · 266/255 · 68/57	MISC. CAT.
23.	12/12 28/27 16/15 4/3	28/27 · 36/35 · 5/4	ARCHYTAS
24.	12/12 16/15 20/18 4/3	16/15 · 25/24 · 6/5	DIDYMOS

4-10. *Summation tetrachords of the second type. Unreduced ratios have been retained to clarify the generating process.*

	TETRACHORD	RATIOS	SOURCE
1.	1/1 10/8 9/7 8/6	5/4 · 36/35 · 28/27	ARCHYTAS
2.	1/1 9/7 17/13 8/6	9/7 · 119/117 · 52/51	MISC. CAT.
3.	1/1 14/11 13/10 12/9	14/11 · 143/140 · 40/39	MISC. CAT.
4.	1/1 13/10 25/19 12/9	13/10 · 250/247 · 76/75	MISC. CAT.
5.	1/1 18/14 17/13 16/12	9/7 · 119/117 · 52/51	MISC. CAT.
6.	1/1 17/13 33/25 16/12	17/13 · 429/425 · 100/99	MISC. CAT.
7.	1/1 22/17 21/16 20/15	22/17 · 357/352 · 64/63	MISC. CAT.
8.	1/1 21/16 41/31 20/15	21/16 · 656/651 · 124/123	MISC. CAT.
9.	1/1 26/20 25/19 24/18	13/10 · 250/247 · 76/75	MISC. CAT.
10.	1/1 25/19 49/37 24/18	25/19 · 931/925 · 148/147	MISC. CAT.

4-11. *Neo-Aristoxenian genera with constant CI.*

PARTS	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETA
ENHARMONIC			
1.5 + 1.5 + 27	25 + 25 + 450	80/79 · 79/78 · 13/10	80/79 · 79/78 · 13/10
1 + 2 + 27	17 + 33 + 450	120/119 · 119/117 · 13/10	120/119 · 119/117 · 13/1
2 + 2 + 26	33 + 33 + 433	56/55 · 55/54 · 9/7	60/59 · 59/58 · 58/45
2.5 + 2.5 + 25	42 + 42 + 417	44/43 · 43/42 · 14/11	48/47 · 47/46 · 23/18
2 + 3 + 25	33 + 50 + 417	55/54 · 36/35 · 14/11	60/59 · 118/115 · 23/18
2 + 4 + 24	33 + 67 + 400	60/59 · 59/57 · 19/15	60/59 · 59/57 · 19/15
3 + 3 + 24	50 + 50 + 400	40/39 · 39/38 · 19/15	40/39 · 38/39 · 19/15
2 + 5 + 23	33 + 83 + 383	56/55 · 22/21 · 5/4	60/59 · 118/113 · 113/90
3 + 4 + 23	50 + 67 + 383	36/35 · 28/27 · 5/4	40/39 · 117/113 · 113/90
3.5 + 3.5 + 23	58 + 58 + 383	32/31 · 31/30 · 5/4	240/233 · 233/226 · 113/
CHROMATIC			
2 + 6 + 22	33 + 100 + 367	51/50 · 18/17 · 100/81	60/59 · 59/56 · 56/45
8/3 + 16/3 + 22	44 + 89 + 367	40/39 · 21/20 · 26/21	45/44 · 22/21 · 56/45
3 + 5 + 22	50 + 83 + 367	34/33 · 22/21 · 21/17	40/39 · 117/112 · 56/45
4 + 4 + 22	67 + 67 + 367	28/27 · 27/26 · 26/21	30/29 · 29/28 · 56/45
2 + 7 + 21	33 + 117 + 350	56/55 · 15/14 · 11/9	60/59 · 118/111 · 37/30
3 + 6 + 21	50 + 100 + 350	34/33 · 18/17 · 11/9	40/39 · 39/37 · 37/30
4 + 5 + 21	67 + 83 + 350	28/27 · 22/21 · 27/22	30/29 · 116/111 · 37/30
4.5 + 4.5 + 21	75 + 75 + 350	24/23 · 23/22 · 11/9	80/77 · 77/74 · 37/30
2 + 10 + 18	33 + 167 + 300	45/44 · 11/10 · 32/27	60/59 · 59/54 · 6/5
3 + 9 + 18	50 + 150 + 300	33/32 · 12/11 · 32/27	40/39 · 13/12 · 6/5
4 + 8 + 18	67 + 133 + 300	28/27 · 243/224 · 32/27	30/29 · 29/27 · 6/5
4.5 + 7.5 + 18	75 + 125 + 300	25/24 · 27/25 · 32/27	80/77 · 77/72 · 6/5
5 + 7 + 18	83 + 117 + 300	21/20 · 15/14 · 32/27	24/23 · 115/108 · 6/5
6 + 6 + 18	100 + 100 + 300	256/243 · 2187/2048 · 32/27	20/19 · 19/18 · 6/5
DIATONIC			
2 + 13 + 15	33 + 217 + 250	45/44 · 44/39 · 52/45	60/59 · 118/105 · 7/6
3 + 12 + 15	50 + 200 + 250	34/33 · 19/17 · 22/19	40/39 · 39/35 · 7/6
4 + 11 + 15	67 + 183 + 250	27/26 · 10/9 · 52/45	30/29 · 116/105 · 7/6
5 + 10 + 15	83 + 167 + 250	104/99 · 11/10 · 15/13	24/23 · 23/21 · 7/6
6 + 9 + 15	100 + 217 + 250	19/18 · 12/119 · 22/19	20/19 · 38/35 · 7/6
7 + 8 + 15	117 + 217 + 250	104/97 · 97/909 · 15/13	120/113 · 113/105 · 7/6
7.5 + 7.5 + 15	125 125 + 250	15/14 · 14/13 · 52/45	16/15 · 15/14 · 7/6
2 + 16 + 12	33 + 267 + 200	64/63 · 7/6 · 9/8	60/59 · 59/51 · 17/15
3 + 15 + 12	50 + 250 + 200	40/39 · 52/45 · 9/8	40/39 · 39/34 · 17/15
4 + 14 + 12	67 + 233 + 200	28/27 · 8/7 · 9/8	30/29 · 58/51 · 17/15
4.5 + 13.5 + 12	75 + 225 + 200	24/23 · 92/81 · 9/8	80/77 · 77/68 · 17/15
5 + 13 + 12	83 + 217 + 200	22/21 · 112/90 · 9/8	24/23 · 115/102 · 17/15
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15
7 + 11 + 12	117 + 183 + 200	16/15 · 10/9 · 9/8	120/113 · 113/102 · 17/1
8 + 10 + 12	113 + 167 + 200	320/297 · 11/10 · 9/8	15/14 · 56/51 · 17/15

Neo-Aristoxenian tetrachords with Ptolemaic interpretations

While Aristoxenos may have been documenting contemporary practice, even a cursory look at his tables suggests that many plausible neo-Aristoxenian genera could be constructed to "fill in the gaps" in his set. The most obvious missing genera are a diatonic with enharmonic diesis, 3 + 15 + 12 (50 + 250 + 200 cents), a *parachromatic*, 5 + 5 + 20 (83 + 83 + 334 cents), and a new soft diatonic, 7.5 + 7.5 + 15 (125 + 125 + 250 cents).

Although Aristoxenos favored genera with 1:1 divisions of the pyknon, Ptolemy and the Islamic writers preferred the 1:2 relation. More complex divisions, of course, are also possible. 4-11 lists a number of neo-Aristoxenian genera in which the CI is held constant and the pyknotic division is varied. With the exception of the first five genera which represent *hyperenharmonic* forms and three which are a closer approximation of the enharmonic (383 cents, rather than 400 cents), only Aristoxenos's CIs are used.

For each tempered genus an approximation in just intonation is selected from a genus in the Main Catalog. Furthermore, an approximation in terms of fractional parts of a string of 120 units of length, analogous to Ptolemy's interpretation of Aristoxenos's genera, is also provided. While these *Ptolemaic interpretations* are occasionally quite close to the ideal tempered forms, they often deviate substantially. One should note, however, that the Ptolemaic approximations are more accurate for the smaller intervals than the larger.

Intervals whose sizes fall between one third and one half of the perfect fourth may be be repeated within the tetrachord, leaving a remainder less than themselves. These are termed reduplicated genera and a representative set of such neo-Aristoxenian tetrachords with reduplication is shown in 4-12.

4-12. *Neo-Aristoxenian genera with reduplication.*

PARTS	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETATION
2 + 14 + 14	34 + 233 + 233	49/48 · 8/7 · 8/7	60/59 · 59/52 · 52/45
4 + 13 + 13	67 + 217 + 217	300/289 · 17/15 · 17/15	30/29 · 116/103 · 103/90
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15
8 + 11 + 11	133 + 183 + 183	27/25 · 10/9 · 10/9	15/14 · 112/101 · 101/90
10 + 10 + 10	166 + 167 + 167	11/10 · 11/10 · 400/363	12/11 · 11/10 · 10/9

4-13. *Neo-Aristoxenian genera with constant pyknotic proportions.*

1:1 PYKNON	CENTS	APPROXIMATION	PTOLEMAIC INTERPRETAT
1.5 + 1.5 + 27	25 + 25 + 450	80/79 · 79/78 · 13/10	80/79 · 79/78 · 13/10
2 + 2 + 26	33 + 33 + 433	56/55 · 55/54 · 9/7	60/59 · 59/58 · 58/45
2.5 + 2.5 + 25	42 + 42 + 417	44/43 · 43/42 · 14/11	48/47 · 47/46 · 23/18
3 + 3 + 24	50 + 50 + 400	40/39 · 39/38 · 19/15	40/39 · 39/38 · 19/15
3.5 + 3.5 + 23	58 + 58 + 383	32/31 · 31/30 · 5/4	240/233 · 233/226 · 113/
4 + 4 + 22	67 + 67 + 367	28/27 · 27/26 · 26/21	30/29 · 29/28 · 56/45
4.5 + 4.5 + 21	75 + 75 + 350	24/23 · 23/22 · 11/9	80/77 · 77/74 · 37/30
5 + 5 + 20	83 + 83 + 334	22/21 · 21/20 · 40/33	24/23 · 23/22 · 11/9
5.5 + 5.5 + 19	92 + 92 + 317	20/19 · 19/18 · 6/5	240/229 · 229/218 · 109/
6 + 6 + 18	100 + 100 + 300	18/17 · 17/16 · 32/27	20/19 · 19/18 · 6/5
6.5 + 6.5 + 17	108 + 108 + 283	17/16 · 16/15 · 20/17	240/227 · 227/214 · 107/
7 + 7 + 16	117 + 117 + 267	16/15 · 15/14 · 7/6	120/113 · 113/106 · 53/4
7.5 + 7.5 + 15	125 + 125 + 250	15/14 · 14/13 · 52/45	16/15 · 15/14 · 7/6
8 + 8 + 14	133 + 133 + 234	14/13 · 13/12 · 7/6	15/14 · 14/13 · 52/45
8.5 + 8.5 + 13	142 + 142 + 217	40/37 · 37/34 · 17/15	240/223 · 223/206 · 103/
9 + 9 + 12	150 + 150 + 200	64/59 · 59/54 · 9/8	40/37 · 37/34 · 17/15
9.5 + 9.5 + 11	158 + 158 + 183	12/11 · 11/10 · 10/9	240/221 · 221/202 · 101/
10 + 10 + 10	166 + 166 + 167	11/10 · 11/10 · 400/363	12/11 · 11/10 · 10/9
1:2 PYKNON			
1 + 2 + 27	17 + 33 + 450	120/119 · 119/117 · 13/10	120/119 · 119/117 · 13/1
4/3 + 8/3 + 26	22 + 44 + 433	84/83 · 83/81 · 9/7	90/89 · 89/87 · 58/45
5/3 + 10/3 + 25	28 + 56 + 417	64/63 · 33/32 · 14/11	72/71 · 71/69 · 23/18
2 + 4 + 24	33 + 67 + 400	57/56 · 28/27 · 24/19	60/59 · 59/57 · 19/15
7/3 + 14/3 + 23	39 + 78 + 383	46/45 · 24/23 · 5/4	360/353 · 353/339 · 113/
8/3 + 16/3 + 22	44 + 89 + 367	40/39 · 21/20 · 26/21	45/44 · 22/21 · 56/45
3 + 6 + 21	50 + 100 + 350	34/33 · 18/17 · 11/9	40/39 · 39/37 · 37/30
10/3 + 20/3 + 20	56 + 111 + 333	33/32 · 16/15 · 40/33	36/35 · 35/33 · 11/9
11/3 + 22/3 + 19	61 + 122 + 317	28/27 · 15/14 · 6/5	360/349 · 349/327 · 109/
4 + 8 + 18	67 + 133 + 300	27/26 · 13/12 · 32/27	30/29 · 29/27 · 6/5
13/3 + 26/3 + 17	72 + 144 + 283	51/49 · 49/45 · 20/17	360/347 · 347/321 · 107/
14/3 + 28/3 + 16	78 + 156 + 267	22/21 · 12/11 · 7/6	180/173 · 173/159 · 53/4
5 + 10 + 15	83 + 167 + 250	104/99 · 11/10 · 15/13	24/23 · 23/22 · 7/6
16/3 + 32/3 + 14	89 + 178 + 233	21/20 · 10/9 · 8/7	45/43 · 43/39 · 52/45
17/3 + 34/3 + 13	94 + 189 + 217	20/19 · 19/17 · 20/17	360/343 · 343/309 · 103/
6 + 12 + 12	100 + 200 + 200	256/243 · 9/8 · 9/8	20/19 · 19/17 · 17/15

Finally, in 4-13, the pyknotic proportions are kept constant at either 1:1 or 1:2 and the CIs are allowed to vary.

These neo-Aristoxenian tetrachords may be approximated in just intonation or realized in equal temperaments whose cardinalities are zero modulo 12. The zero modulo 12 temperaments provide opportunities to simulate many of the other genera in the Catalogs as their fourths are only two cents from 4/3 and other intervals of just intonation are often closely approximated. One may also use them to discover or invent new neo-Aristoxenian tetrachords.

To articulate a single part difference, a temperament of 72 tones per octave is required. The 1/2 parts in the hemiolic chromatic and several other genera normally demand 144 tones unless all the intervals including the disjunctive tone have a common factor. In this case, the 48-tone system suffices. For the 1:2 pykna which employ 1/3 parts, 216-tone temperament is necessary unless the numbers of parts share common factors. These data are summarized in 4-14.

4-14. *Aristoxenian realizations. The framework is the number of "parts" in the two tetrachords and the disjunctive tone. The corresponding equal temperament is the sum of the parts of the framework. The articulated genera are those that may be played in the corresponding equal temperaments. The scheme of 144 parts was used by Avicenna and Al-Farabi (D'Erlanger 1930).*

FRAMEWORK	ET	ARTICULATED GENERA
5 2 5	12	*Diatonic and syntonic chromatic.*
10 4 10	24	*Enharmonic, syntonic and soft diatonics, syntonic chromatic.*
15 6 15	36	*Syntonic diatonic, syntonic and soft chromatics, unnamed.*
		Chromatic, diatonic with soft chromatic dieses.
20 8 20	48	*Hemiolic chromatic, soft and syntonic diatonics, syntonic chromatic, diatonic with hemiolic chromatic dieses. See 24-tone ET.*
25 10 25	60	*Syntonic diatonic and chromatic.*
30 12 30	72	*All previous genera except hemiolic chromatic and genera with hemiolic chromatic dieses (see 24-tone ET).*
35 14 35	84	*Syntonic diatonic and chromatic.*
40 16 40	96	*Enharmonic, syntonic diatonic, soft diatonic, syntonic and hemiolic chromatic. See 24-tone ET.*
45 18 45	108	*See 36–tone ET.*
50 20 50	120	*See 24-tone ET.*
55 22 55	132	*See 12–tone ET.*
60 24 60	144	*All genera except 1:2 pykna with 1/3 parts.*
90 36 90	216	*All genera defined in text.*

Semi-tempered tetrachords

The computation of the mean tetrachords also generates a number of genera containing irrational intervals involving square roots. These tetrachords contain both tempered intervals as well as at least one in just intonation, the 4/3, and may therefore be called *semi-tempered.* There also are the semi-tempered tetrachords resulting from a literal interpretation of the late classical theorists Nichomachos and Thrasyllus (Barbera 1978). The first of these is Nichomachos's enharmonic, defined verbally as a ditone with an equally divided *limma* and mathematically as $\sqrt{(256/243)} \cdot \sqrt{(256/243)} \cdot 81/64$ (45 + 45 + 408 cents). The second is Thrasyllus's chromatic, described analogously as having a Pythagorean *trihemitone* or minor third and a whole tone pyknon. Literally, this genus would be $\sqrt{(9/8)} \cdot \sqrt{(9/8)} \cdot 32/27$ (102 + 102 + 294 cents), but it is possible that Thrasyllus meant the standard Pythagorean tuning in which the pyknon consists of a limma plus an *apotome*, i.e., 256/243 · 2187/2048 · 32/27 (90 + 114 + 294 cents).

Other semi-tempered forms result from Barbera's assumption that Aristoxenos may have intended that the perfect fourth of ratio 4/3 be divided geometrically into thirty parts. Barbera (1978) offers this literal version of the enharmonic: $\sqrt[10]{(4/3)} \cdot \sqrt[10]{(4/3)} \cdot \sqrt[10]{(65536/6561)}$, or 50 + 50 + 398 cents, where 65536/6561 is $(4/3)^8$. It is an easy problem to find analogous interpretations of the remainder of Aristoxenos's genera. These and a few closely related genera from 3-1–3 have been tabulated in 4-15.

4-15. *Semi-tempered Aristoxenian tetrachords. These tetrachords are literal interpretations of Aristoxenos's genera under Barbera's assumption that Aristoxenos meant to divide the perfect fourth of ratio 4/3 into 30 equal parts.*

	PARTS	ROOTS	CENTS	GENUS
1.	3 + 3 + 24	$4/3^{1/10} \cdot 4/3^{1/10} \cdot 4/3^{4/5}$	50 + 50 + 398	ENHARMONIC
2.	4 + 4 + 22	$4/3^{2/15} \cdot 4/3^{2/15} \cdot 4/3^{11/15}$	66 + 66 + 365	SOFT CHROMATIC
3.	4.5 + 4.5 + 21	$4/3^{3/20} \cdot 4/3^{3/20} \cdot 4/3^{7/10}$	75 + 75 + 349	HEMIOLIC CHROMATIC
4.	6 + 6 + 18	$4/3^{1/5} \cdot 4/3^{1/5} \cdot 4/3^{3/5}$	100 + 100 + 299	INTENSE CHROMATIC
5.	6 + 9 + 15	$4/3^{1/5} \cdot 4/3^{3/10} \cdot 4/3^{1/2}$	100 + 149 + 250	SOFT DIATONIC
6.	6 + 12 + 12	$4/3^{1/5} \cdot 4/3^{2/5} \cdot 4/3^{2/5}$	100 + 199 + 199	INTENSE DIATONIC
7.	4 + 14 + 12	$4/3^{2/15} \cdot 4/3^{7/15} \cdot 4/3^{2/5}$	66 + 232 + 199	DIATONIC WITH SOFT CHROMATIC DIESES
8.	4.5 + 13.5 + 12	$4/3^{3/20} \cdot 4/3^{9/20} \cdot 4/3^{2/5}$	75 + 224 + 199	DIATONIC WITH HEMIOLIC CHROMATIC DIESES
9.	4 + 8 + 18	$4/3^{2/15} \cdot 4/3^{4/15} \cdot 4/3^{3/5}$	66 + 133 + 299	UNNAMED
10.	6 + 3 + 21	$4/3^{1/5} \cdot 4/3^{1/10} \cdot 4/3^{7/10}$	100 + 50 + 349	REJECTED
11.	4.5 + 3.5 + 22	$4/3^{3/20} \cdot 4/3^{7/60} \cdot 4/3^{11/15}$	75 + 58 + 365	REJECTED
12.	10 + 10 + 10	$4/3^{1/3} \cdot 4/3^{1/3} \cdot 4/3^{1/3}$	166 + 166 + 166	SEMI-TEMPERED EQUABLE DIATONIC
13.	12 + 9 + 9	$4/3^{2/5} \cdot 4/3^{3/10} \cdot 4/3^{3/10}$	200 + 149 + 149	ISLAMIC DIATONIC

Equal divisions of the 4/3

The semi-tempered tetrachords suggest that equally tempered divisions of the 4/3 would be worth exploring. Such scales would be analogous to the equal temperaments of the octave except that the interval of equivalence is the 4/3 rather than the 2/1. Scales of this type are very rare, though they have been reported to exist in contemporary Greek Orthodox liturgical music (Xenakis 1971).

A possible ancestor of such scales is the ancient Lesser Perfect System, which consisted of a chain of the three tetrachords hypaton, meson, and synemmenon. In theory, all three tetrachords were identical, but this was not an absolute requirement, and in fact, in Ptolemy's mixed tunings, they would not have been the same. (See chapter 6 for the derivations of the various scales and systems, and chapter 5 for the analysis of their properties.)

The most interesting equal divisions of the 4/3 resemble the equal temperaments described in the next section and in 4-14 and 4-17. The melodic possibilities of these scales should be quite rich, because in those divisions with more than three degrees to the 4/3 not only can several tetrachordal genera be constructed, but various permutations of these genera are also possible.

The harmonic properties, however, may be very different from those of the octave divisions as the 2/1 may not be approximated closely enough for octave equivalence to be retained. Moreover, depending upon the division, other intervals such as the 3/2 or 3/1 may or may not be acceptably consonant.

The equal divisions of the 4/3 which correspond to equal octaval temperaments are described in 4-16. A few supplementary divisions such as the one of 11 degrees have been added since they reasonably approximate harmonically important intervals. For reasons of space, only a very limited number of intervals was examined and tabulated. To gain an adequate understanding of these tunings, the whole gamut should be examined over a span of at least eight 4/3's.

Additionally, the nearest approximations to the octave and the number of degrees per 2/1 are listed. This information allows one to decide whether the tuning is equivalent to an octave division, or whether it essentially lacks octave equivalence. Composition in scales without octave equivalence is a relatively unexplored area, although the

DEGREES PER 4/3	CENTS/DEGREE	DEGREES/OCTAVE	CENTS/OCTAVE	OCTAVE DIVISION	OTHER CONSONANT INTERVA
3	166.0	7.228	1162.1	7 (–)	GOLDEN RATIO (PHI) = 5
4	124.5	9.638	1245.1	10 (+)	7/1 = 27
5	99.61	12.05	1195.3	12 (–)	5/1 = 28
6	83.01	14.46	1162.1	14 (–)	7/5 = 7
7	71.15	16.86	1209.5	17 (+)	—
8	62.26	19.27	1182.9	19 (–)	7/1 = 54
9	55.34	21.68	1217.4	22 (+)	5/3 = 16, 6/1 = 56
10	49.80	24.09	1195.3	24 (–)	3/2 = 14, 5/1 = 56
11	45.28	26.50	1222.5	27 (+)	3/1 = 42, 4/1 = 53, 5/2 = 35,
13	38.31	31.32	1187.6	31 (–)	6/1 = 81, 7/1 = 88, 8/1 = 94
14	35.57	33.73	1209.5	34 (+)	7/2 = 61
15	33.20	36.14	1195.3	36 (–)	5/1 = 84, PHI = 25
17	29.30	40.96	1201.2	41 (+)	3/2 = 24, 7/2 = 74
20	24.90	48.19	1195.3	48 (–)	5/1 = 112, 7/4 = 39
22	22.64	53.01	1199.8	53 (–)	3/2 = 31, 5/3 = 39
25	19.92	60.24	1195.3	60 (–)	5/1 = 140, 7/1 = 169
28	17.79	67.46	1191.8	67 (–)	3/1 = 107, 4/1 = 135
30	16.605	72.28	1195.3	72 (–)	7/1 = 203, 7/5 = 35
35	14.23	84.33	1195.3	84 (–)	7/4 = 68, 7/5 = 41
40	12.45	96.38	1195.3	96 (–)	6/1 = 249, 5/3 = 71
45	11.07	108.4	1195.3	108 (–)	3/1 = 172, 4/1 = 217
50	9.961	120.5	1195.3	120 (–)	3/1 = 191, 4/1 = 241
55	9.055	132.5	1204.4	133 (+)	7/4 = 107, PHI = 92, 3/1 = 21
60	8.301	144.6	1203.6	145 (+)	3/1 = 229, 4/1 = 289
90	5.534	216.8	1200.8	217 (+)	3/2 = 127

4-16. *Equal divisions of the 4/3. These are equal temperaments of the 4/3 rather than the 2/1. "Degrees/octave" is the number of degrees of the division corresponding to the 2/1 or octave. For many of these divisions, the octave no longer functions as an interval of equivalence. "Cents/octave" is the cent value of the approximations to the 2/1. "Octave division" is the closest whole number of degrees to the 2/1. (–) indicates that the octave is compressed and less than 1200 cents. (+) means that it is stretched and larger than 1200 cents. "Consonant intervals" are the degrees in good approximations to the intervals listed. All divisions of the 4/3 have good approximations to the 10/1 as* $(4/3)^8$ *+ the skhisma equals 10/1. Divisons that are multiples of 3 also have good approximations to the 11/1. 17 is a slightly stretched 41-tone equal temperament. 22 is audibly equivalent to 53-tone equal temperament. 28 is analogous to the division of the fourth into 28 parts according to Tiby's theory of Greek Orthodox liturgical music (Tiby 1938). 30 is analogous to Aristoxenos's basic system. 55 is analogous to 132-tone equal temperament. 60 is analogous to 144-tone equal temperament. 90 is analogous to 216-tone equal temperament. The Golden Ratio or Phi is* $(1+\sqrt{5})/2$*, approximately 1.618.*

composer and theorist Brian McLaren has recently written a number of pieces in non-octaval scales mostly of his own invention (McLaren, personal communication, 1991). Xenakis has also mentioned chains of fifths consisting of tetrachords and disjunctive tones (Xenakis 1971). These suggest analogous divisions of the 3/2, including both those with good approximations to the 4/3 and those without. Similarly, there are divisions in which octave equivalence is retained and those in which it is not. An example of one with both good fourths and octaves is the seventh root of 3/2, which corresponds to a moderately stretched 12-tone equal temperament of the octave (Kolinsky 1959).

Tetrachords in non-zero modulo 12 equal temperaments

Tetrachords may also be defined in non-zero modulo 12 equal temperaments. For some combinations of genus and tuning the melodic and harmonic distortions will be negligible, but for others the mappings may distort the characteristic melodic shapes unacceptably. As an illustration, the three primary genera, the enharmonic, the syntonic chromatic, and the

4-17. *Tetrachords in non-zero modulo 12 equal temperaments. These genera are defined in ETs where the perfect fourth does not equal 2 1/2 "whole tones." The framework is the number of "parts" in the two fourths and the disjunctive tone. More than one framework is plausible in some temperaments without good fourths or with more than 17 notes. The corresponding equal temperament is the sum of the parts of the framework. The genera in a generalized, non-specific sense may be approximated in these equal temperaments. "Diatonic/chromatic" means that there is no melodic distinction between these genera. The chromatic pykna in 9-, 10-, and 11- tone ET consist of two small intervals and one large, while the disjunction may larger or smaller than the CI. Genera indifferently enharmonic and chromatic occur around 19 tones per octave and neo-Aristoxenian forms may be realizable in many of the ETs.*

FRAMEWORK	ET	GENERA
3 1 3	7	DIATONIC/CHROMATIC
3 2 3	8	DIATONIC/CHROMATIC
4 1 4	9	CHROMATIC
4 2 4	10	CHROMATIC
4 3 4	11	CHROMATIC
5 3 5	13	DIATONIC, CHROMATIC
6 2 6	14	DIATONIC, CHROMATIC
6 3 6	15	DIATONIC, CHROMATIC
7 2 7	16	DIATONIC, CHROMATIC
7 3 7	17	DIATONIC, CHROMATIC
7 4 7 (8 2 8)	18	DIATONIC, CHROMATIC (ALL THREE)
8 3 8	19	DIATONIC, CHROMATIC
8 4 8	20	ALL THREE
9 3 9, 8 5 8	21	ALL THREE
9 4 9	22	ALL THREE
9 5 9, 10 3 10	23	ALL THREE
13 5 13	31	ALL THREE
14 6 14	34	ALL THREE
17 7 17	41	ALL THREE
22 9 22	53	ALL THREE

4-18. Augmented and diminished tetrachords. These tetrachords are closely related to those in 8-5 and 8-15. For tetrachords with perfect fourths incorporating the diminished fourths as intervals, see the Main and Miscellaneous Catalogs. A few additional intervals of similar size have been used as CIs in 4-1, but not divided due to their complexity. The last three intervals are technically diminished fifths, but they function as augmented fourths in certain of the harmoniai of chapter 8.

RATIOS	CENTS	EXAMPLES
14/11	418	14/13 · 13/12 · 12/11
23/18	424	23/22 · 11/10 · 10/9
32/25	427	32/31 · 31/30 · 6/5
9/7	435	18/17 · 17/16 · 8/7
31/24	443	31/30 · 10/9 · 9/8
22/17	446	11/10 · 10/9 · 18/17
13/10	454	13/12 · 12/11 · 11/10
30/23	460	15/14 · 7/6 · 24/23
17/13	464	17/16 · 8/7 · 14/13
21/16	471	21/20 · 10/9 · 9/8
29/22	478	29/28 · 7/6 · 12/11
31/23	517	31/30 · 5/4 · 24/23
23/17	523	23/22 · 11/9 · 18/17
19/14	529	19/18 · 6/5 · 15/14
15/11	537	15/14 · 7/6 · 12/11
26/19	543	26/25 · 5/4 · 20/19
11/8	551	11/10 · 10/9 · 9/8
40/29	557	8/7 · 7/6 · 30/29
18/13	563	9/8 · 8/7 · 14/13
25/18	569	5/4 · 20/19 · 19/18
32/23	572	16/15 · 5/4 · 24/23
7/5	583	14/13 · 13/12 · 6/5
1024/729	588	256/243 · 8/7 · 7/6
45/32	590	16/15 · 10/9 · 6/5
24/17	597	6/5 · 10/9 · 18/17
17/12	603	17/16 · 8/7 · 7/6
44/31	606	11/10 · 5/4 · 32/31
10/7	617	10/9 · 9/8 · 8/7

diatonic, will be mapped into the 12-, 19-, 22-, and 24-tone equal temperament (ET) below:

ET	FOURTH	ENHARMONIC	CHROMATIC	DIATONIC
12	5°	—	1 + 1 + 3	1 + 2 + 2
19	8°	1 + 1 + 6	2 + 2 + 4	2 + 3 + 3
22	9°	1 + 1 + 7	2 + 2 + 5	1 + 4 + 4
24	10°	1 + 1 + 8	2 + 2 + 6	2 + 4 + 4

The enharmonic is not articulated in 12-tone ET, or at least not distinguishable from the chromatic except as a semitonal-major third pentatonic. In 19-tone ET, the soft chromatic is identical to the enharmonic and the syntonic chromatic is close to a diatonic genus like 125 + 125 + 250 cents. The enharmonic is certainly usable in 22-tone ET but the diatonic is deformed, with a quarter-tone taking the place of the semitone. These distortions, however, are mild compared to the 9-tone equal temperament in which not only are the diatonic and chromatic genera equivalent as 1 + 1 + 2 degrees, but the semitone at two units is larger than the whole tone. Whether these intervallic transmogrifications are musically useful remains to be tested.

There are, however, many fascinating musical resources in these non-12-tone tunings. As Ivor Darreg has pointed out, each of the equal temperaments has its own particular mood which suffuses any scale mapped into it (Darreg 1975). For this reason the effects resulting from transferring between tuning systems may be of considerable interest.

Because of the large number of systems to be covered, the mappings of the primary tetrachordal genera into the non-zero modulo 12 equal temperaments are summarized in 4-17. The tetrachordal framework and primary articulated genera in the equal temperaments of low cardinality or which are reasonable approximations to just intonation are shown in this figure.

Augmented and diminished tetrachords

The modified or altered tetrachords found in some of the non-zero modulo 12 equal temperaments of 4-17 suggest that tetrachords based on augmented and diminished fourths might be musically interesting. This supposition has historical and theoretical support. The basic scales (*thats*) of some Indian ragas have both augmented and perfect fourths (Sachs 1943), and the octaval *harmoniai* of Kathleen Schlesinger contain fourths of dimagnitudes (Schlesinger 1939; and chapter 8). Wilson has exploited the fact that any scale generable by a chain of melodic fourths must incorporate fourths of at least two magnitudes (Wilson 1986; 1987; and chapter 6). His work implies that scales may be produced from chains of fourths of any type, but that their sizes and order must be carefully selected to ensure that the resulting scales are recognizably tetrachordal.

A number of altered fourths are available for experimentation. 4-18 lists those which commonly arise in conventional theory and in the extended theory of Schlesinger's harmoniai described in chapter 8. Scales may be constructed by combining these tetrachords with each other or with normal ones and with correspondingly altered disjunctive tones to complete the octaves. Alternatively, the methods described in chapter 6 to generate non-heptatonic scales may be employed.