

**Pitch, Harmony and Experimental Intonation**  
**A primer**

**Class Notes**  
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## The mechanics of intervals (the language of ratios)

### The harmonic series as intonation “canon”

Many authors, including Partch, cite the harmonic series as either an acoustical or mathematical motivation for simple just intonational systems. The intonations of integers-multiples of a given fundamental, in octave reduced form, form the basis for most (but not all) tuning systems, even in tempered variations.

It is useful to memorize the harmonic series up to at least the first 16–17 harmonics — both the approximate 12TET (12 tone equal-temperament) pitch values and the corresponding cents deviations from equal temperament.

The following are the harmonic numbers with corresponding intervals and cents deviations from 12TET tuning underneath:

<i>Harmonic</i>	1	2	3	4	5	6	7	8	
<i>Interval</i>	uni	oct	5th	oct	3rd	5th	min7	oct	
<i>Cents from 12-ET</i>	+0	+0	+2	+0	-14	+2	-31	+0	
<i>Harmonic</i>	9	10	11	12	13	14	15	16	17
<i>Interval</i>	9th	3rd	tt	5th	min6	min7	maj7	oct	min2nd
<i>Cents from 12-ET</i>	+4	-14	-49	+2	+41	-31	-12	+0	+5

The “names” of the intervals (5<sup>th</sup>, 3<sup>rd</sup>, etc.) especially in the case of the 11th and 13th harmonic, are approximations of some nearby, cognate 12TET interval.

### Expressing a ratio between 1/1 and 2/1 (within one octave)

Often (but not always), just intonation scale systems are *octave reduced*, and specified in simple form (numerator and denominator reduced by common factors). That is, ratios are most often expressed in the simplest form of the fraction, with the numerator and denominator relatively prime. Thus, 9/3 becomes 3/1, 10/6 becomes 5/3, and so on.

But powers of 2 are also special, because they indicate octave relationships. Powers of 2 are often a factor of the numerator and denominator, and they can be changed without changing the “interval class.” We often want to state a ratio, or interval, to be within an octave, in the following way:

$$1/1 \geq p/q \leq 2/1$$

A simple procedure for reducing a ratio to its “smaller than octave” form is to multiply either the numerator or denominator by two until the fraction is within range, and then reduce the fraction to simplest form. Thus, 16/5 becomes 8/5, 12/3 becomes 4/3, and so on.

Here’s an example of how to express a ratio to its simplest, octave reduced form:

**Example:** 8/3

- Step 1: Is the numerator more than twice the denominator? (Yes): Multiply the denominator by 2 until it the numerator is less than twice the denominator (8/6).
- Step 2: Reduce by common factors until the numbers are relatively prime (4/3)

**Example:** 15/16

- Step 1: Is the numerator less than the denominator? (Yes): Either divide the denominator by 2 (if possible), or multiply the numerator by 2: (15/8)
- Step 2: Reduce by common factors until the numbers are relatively prime: (15/8)

Note that the octave reduction process is equivalent to moving a ratio along the “2-axis” in harmonic space (in Tenney’s or Johnston’s terms). For this reason (something like “octave equivalence”), that axis is often omitted in representations. This is similar to talking about a P5<sup>th</sup> as being the same for many purposes as a P12<sup>th</sup>.

### **Multiplying two ratios (adding two intervals)**

What is, in 12TET, a M3<sup>rd</sup> above a P5<sup>th</sup>? To answer that, we “add” the intervals and come up with a M7<sup>th</sup>.

To add two ratios we multiply them together. In order to find out what is a P5<sup>th</sup> above a M3<sup>rd</sup>, we add the two intervals, multiplying their ratios. For example, to determine a *septimal* (has a factor of 7 in it) (7/6) m3<sup>rd</sup> above a P5<sup>th</sup> (3/2), multiply the two ratios, and then reduce to simple form (including, usually, octave reducing):

**Example:**  $7/6 * 3/2$

(a septimal m3<sup>rd</sup> above a P5<sup>th</sup>)

Multiply the ratios together, reduce to relatively prime form, and then to within one octave:

$$21/12 = 7/4 \text{ (septimal m7}^{\text{th}} \text{) (-31 from 12TET)}$$

**Example:**  $9/5 * 5/3$

(just m7<sup>th</sup> above a just M6<sup>th</sup>)

$$45/15 = 3/1 = 3/2 \text{ (P5}^{\text{th}} \text{) (+2 from 12TET)}$$

**Example:**  $27/16 * 3/2$

(Pythagorean M6<sup>th</sup> above a just or Pythagorean P5<sup>th</sup>)

$$81/32 = 81/64 \text{ (Pythagorean M3}^{\text{rd}} \text{) (+8 from 12TET, +22 from 5/4 just M3}^{\text{rd}} \text{, or the “comma”)}$$

[Note: By “Pythagorean,” we mean only the prime factor 3 (and of course, 2) is involved: all tuning is done by P5<sup>ths</sup>.]

### **Dividing two ratios (subtracting one interval from another)**

To get a m7<sup>th</sup> below a M3<sup>rd</sup> in 12TET, we subtract one interval from the other (and get the tritone).

To subtract two ratios, we divide them. In order to find out what is a P5<sup>th</sup> below a M3<sup>rd</sup>, we subtract the two intervals, dividing their ratios. For example, to determine a septimal M2<sup>nd</sup> (8/7) below a just M7<sup>th</sup> (15/8), divide the two ratios, and then reduce to simple form (including, usually, octave reducing).

Note that arithmetically, dividing one ratio by another is the same as inverting the second ratio and multiplying:

**Example:**  $15/8 \ / \ 8/7$   
(a large M2<sup>nd</sup> below a M7<sup>th</sup>)  
Invert the second ratio, multiply, and reduce as above  
 $15/8 * 7/8 = 105/64$  M6<sup>th</sup> (-19 from 12TET)

Note: This is a “narrow” M6<sup>th</sup>, because 8/7 is a wide M2<sup>nd</sup>.

### **Determining the cents (¢) value of an interval, and determining what that interval “is” (quick and dirty method)**

Cents are a way to compare all intervals, just or not. 100 cents (100¢) is a 12TET m2<sup>nd</sup>. 1200¢ is an octave. Although there is, of course, a kind of “western” bias in this choice of measurement, it can also be thought of as an aesthetically neutral system that simply determines the width of an interval in pitch (not frequency) space, and allows us to compare two intervals.

*Method 1* (expert system approach): Go on the web, use a simple computer program, consult either the Partch table at the end of *Genesis ... the Chalmers “Table of 1200 Tone Equal Temperament,”* or any one of a number of such charts.

The cents value of a frequency ratio is:  
$$\text{¢} = 1200 * \log_2(a/b)$$

— where  $r$  is some ratio, generally in reduced form. For convenience, cents values are often rounded to the nearest cent. For example,

$$3/2 = 702\text{¢}$$

*Method 2* (wetware-based artificial intelligence approach): Doing it quickly in your head (amaze your friends):

**Example:** 75/54 (what is it? how wide is it?)  
*Step 1:* Reduce the numerator and denominator to relative prime form, and then determine the prime factors:

$$\begin{aligned} 75 &= 5 * 5 * 3 \\ 54 &= 3 * 3 * 3 * 2 \end{aligned}$$

One set of three's cancel, so  $75/54 = 25/18$ . You could (and should) have done that first. Thus we have  $5 * 5$  on top, and  $3 * 3 * 2$  on the bottom.

*Step 2:* Add up the cents deviations for each prime from the harmonic series in the numerator.

$$-14 + -14 = -28$$

*Step 3:* Add up the cents deviations for each prime from the harmonic series in the denominator, *reversing their sign*.

$$-2 + -2 + 0 = -4$$

*Step 4:* Determine, approximately, what "note" the numerator refers to (simply by associating simple intervals with primes, and let's assume  $1/1 = C$  for the purposes of this example):

$$5 \times 5 = \text{"third"} \text{ above a "third"} = +5^{\text{th}} = G\#$$

*Step 5:* Determine, approximately, what "note" the denominator refers to:

$$3 \times 3 \text{ two fifths} = M2^{\text{nd}} = D$$

*Step 6:* What "interval" does the whole ratio (approximately) refer to:

$$+ 5^{\text{th}} \text{ above a } M2^{\text{nd}} = \text{tritone (D-G\#)}$$

*Step 7:* To get the cents deviation from the nearest 12TET interval, add the cents deviation of the numerator to the (sign reversed) denominator:

$$-28 + -4 = -32$$

Therefore, the ratio  $75/64$  is a tritone  $-32\text{c}$  flat of 12TET (Note that simply knowing, for instance, with the note "C" as the fundamental, the 25th harmonic is a  $+5^{\text{th}}$  and the 18th harmonic is a  $M2^{\text{nd}}$ , give you the interval right away!).

*Digression 1:* An interesting method for learning tunings was suggested (and often used) by Lou Harrison: using a simple just intonation as the canonical tuning from which to deviate. In other words, the Pythagorean  $27/16$  major  $6^{\text{th}}$ , which is  $6\text{c}$  sharp of 12TET, would be considered  $+22\text{c}$  sharp of the "canonical"  $5/3$  (just, 5-limit) major  $6^{\text{th}}$  (the famous commatic difference between just and Pythagorean tuning).

*Digression 2:* A simple vector notation for intervals is, for example, [002] for 25, [12] for 18, where each place in the vector is a count of the number of occurrences of a prime (the exponent of each prime in the factorization), starting with 2, then 3, 5, 7, 11, ... To represent a ratio in this way, use minuses for occurrence of factors in the denominator, positives for the numerator. Thus  $25/18 = [-1-22]$ . This is a computationally valuable and unique vector representation of all fractions, and, while equivalent to other forms, allows for some interesting manipulation in determining things like “consonance” and “dissonance.”

## Some common scales, tuning schemes and principles

### A simple Pythagorean sequence

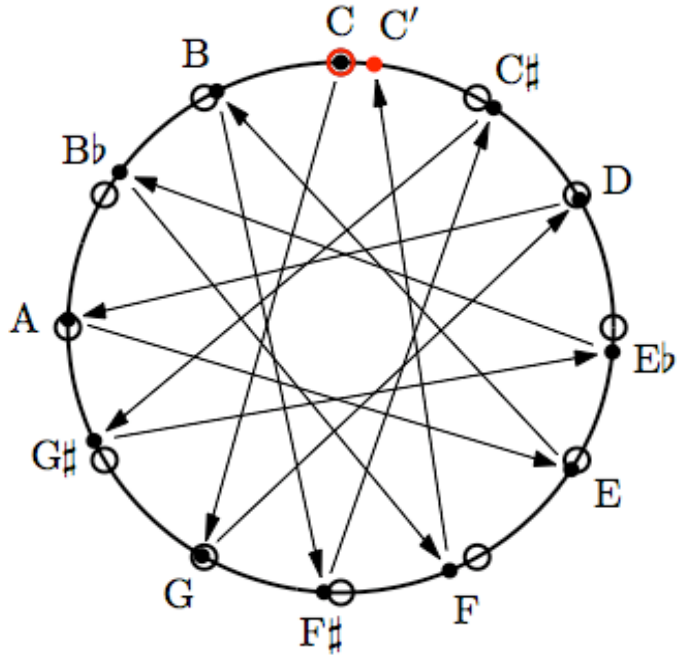
Pythagorean tunings use only ratios involving 3 and 2. Partch refers to them as 3-limit tunings, Tenney/Johnston call them as 2,3-space. Their “prime vectors” only have two places.

Expressed in the key of C:

$$1/1(C+0) \quad 3/2(G+2) \quad 9/8(D+4) \quad 27/16(A+6)$$

$$81/64(E+8) \quad 243/128(B+10) \quad 729/512(F\#+12) \dots$$

Note that this sequence is “cyclic,” no sequence of 3/2’s ever yields a 2/1, so Pythagorean tunings must somehow adjust for the octave (another type of comma, the Pythagorean one, the distance between 2/1 and 531441/524288, about 24¢, or the sum of 12 3/2’s (each adds 2¢).



**THE PYTHAGOREAN COMMA**

The Pythagorean comma results from the “circle of fifths,” when those intervals are tuned as the ratio  $3/2$ . Compounding  $5^{\text{th}}$ s (C-G-D-A-E-B-F#-C#-G#-D#-A#-F(E#)-C) will never result in an in-tune octave ( $2/1$ ). This is the simplest example of the “historical tuning problem.” In the illustration above the difference between the compounded  $3/2$   $5^{\text{th}}$ s are solid dots, connected by arrows showing the direction of the tuning. Open circles are the corresponding equal-tempered  $5^{\text{th}}$ s. Each  $3/2$   $5^{\text{th}}$  adds  $2\text{c}$  to the difference between these intervals, culminating in the  $24\text{c}$  comma shown by the red dot when the tuning comes full circle.

Remember again, no number raised to a prime power can ever equal another number raised to a different prime power (The Diophantine equation:  $p^n = q^m$  for any two primes  $p, q$ , and any integers  $n, m$ ). Any system which includes more than one prime (even, as in Pythagorean tuning, 2 and 3!) will never be “in tune” in that certain simple intervals will always “conflict” with other simple intervals. If that conflict occurs in the simple ( $\{3, 5\}$ , or “5-limit” just scale, it’s usually called the “wolf fifth.” It is a more general phenomenon, however. I call it the “canidae” interval (genus, not species). It is also sometimes referred to, informally, as the “historical tuning problem.”

**The Syntonic comma**

The difference between the  $5/4$  M3<sup>rd</sup> and the  $81/64$  Pythagorean M3<sup>rd</sup> is one of the most important engines in the evolution of tuning systems. Put simply, it’s hard to have (simultaneously) good 5<sup>th</sup>s and good 3<sup>rd</sup>s (see the section on the wolf-fifth below) if only a rational tuning is employed, because of the conflict of primes. Comma relationships (the syntonic comma is *the* comma) are often notated with a “-” or “+” sign (as in Ben Johnston’s notational system):

$$81/64 / 5/4 = 81/80 = 22\text{c (app.)}$$

**A standard Just diatonic (absolute ratios); 5-limit (Ptolemy's Syntonon Diatonic)**

With cents deviations from 12TET underneath each ratio:

1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
+0	+4	-14	-2	+2	-16	-12	+0

This scale is often simply referred to as the “just scale,” though  $10/9$  may often be substituted for  $9/8$  in simple just scales.

**A simple just diatonic (adjacent ratios); 5-limit (adjacency ratios for Syntonon Diatonic)**

1/1	9/8	10/9	16/15	9/8	10/9	9/8	16/15
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This scale is the same as the one above, but expressed as *adjacency intervals* rather than as ratios from 1/1. That is, we are describing the scale as “M2<sup>nd</sup> M2<sup>nd</sup> m2<sup>nd</sup> M2<sup>nd</sup> ....” Instead of “M2<sup>nd</sup> M3<sup>rd</sup> P4<sup>th</sup> P5<sup>th</sup> ...” As in 12TET we can reckon intervals to other intervals, or to some fixed 1/1 (fundamental). We freely go between those representations, depending on context.

Using adjacency intervals emphasizes the melodic. Lou Harrison often preferred this method of elucidating scales, and uses it in for the majority of the scales in his *Music Primer* (as well as his remarkable slendro and pelog tunings in Gamelan Si Darius/Si Madeleine). It also more closely represents his idea of “free-style.”



*Si Darius and Si Madeleine, Mills College, 1981*

(Adjacent intervals, ratios, and cents values of intervals)

slendro (Darius)

5-6 (8:7) 6-1 (7:6) 1-2 (8:7) 2-3 (8:7) 3-5 (7:6)  
231¢ 267¢ 204¢ 231¢ 267¢

pelog (Madeleine):

1-2 (13:12) 2-3 (14:13) 3-4 (17:14) 4-5(18:17) 5-6(19:18) 6-7 (21:19) 7-1 (8:7)  
139¢ 128¢ 336¢ 99¢ 94¢ 173¢ 231¢

**The wolf-fifth (on the supertonic)**

The usual example for the “historical tuning problem” in just intonation involves the problem between tuning scale degrees 2, 3, and 6, since there is a 5<sup>th</sup> between 2 and 6, and from 6 up to 3. The problem is the basic conflict between 3- and 5-limit tuning systems, and could be said to have been the motivation for the long history of “compromise” tunings (meantones, well-temperaments, and finally, 12TET). These tunings “temper” interval so as to distribute the comma, or error.

There are many other possible instances of this kind of cross-tuning difficulty. The problem is the “collision” between scale systems of different limits, not just 3- and 5-. In much of twentieth-century music, like that of James Tenney, Carter Scholz, and others, this has become a feature rather than a bug.

The following is a simple explanation of the wolf-fifth:

If the 2<sup>nd</sup> of a simple diatonic just scale is tuned as 9/8, and the 6<sup>th</sup> as 5/3, then  $5/3 \div 9/8 = 40/27$  (20 cents flat of a 3/2!), the infamous wolf-fifth (on the supertonic). That is, the P5<sup>th</sup> between the 2<sup>nd</sup> and 6<sup>th</sup> degrees, necessary, for example, in the supertonic triad, is “out of tune.” However, the 5<sup>th</sup> from a 5/3 up to 5/4, on the submediant, is “good” (read: 3/2). Conversely, if the 6<sup>th</sup> degree is tuned to Pythagorean 27/16, and the 3<sup>rd</sup> to 5/4, then the same situation arises in the 5<sup>th</sup> above the submediant ( $27/16 \div 5/4 = 27/20$ ), the P4<sup>th</sup> which is the inversion of the 40/27).

	C	D	E	F	G	A	B	C
	(1/1)	(9/8)	(5/4)	(4/3)	(3/2)	(5/3)	(15/8)	(2/1)
C		9/8	5/4	4/3	<b>3/2</b>	5/3	15/8	2/1
D			10/9	32/27	4/3	<b>40/27</b>	35/18	16/9
E				16/15	6/5	4/3	<b>3/2</b>	8/5
F					9/8	5/4	45/32	<b>3/2</b>
G						10/9	5/4	4/3
A							9/8	6/5
B								16/15

**½ Matrix of Diatonic Just Intonation Scale**

Every scale can be represented as a matrix of relationships between each degree of the scale and all others. This half-matrix shows those relationships in the historic Just Diatonic scale described above. Intervals are given as ratios between each interval and all others. 5<sup>th</sup>s are in bold. The “wolfP” 5<sup>th</sup> is the interval 40/27 between the 2<sup>nd</sup> and 6<sup>th</sup> scale degree — the only “impure” P5<sup>th</sup> in the scale. Note that all M3<sup>rd</sup>s (C-E, F-A, G-B) are “pure” intervals of 5/4.

**A simple just chromatic (adjacent ratios); 5-limit**

Much of the following scale is derived from intervallic inversions of the standard 5-limit just scale. For example, 16/15 (the m2<sup>nd</sup> to the tonic) is the same as the interval between the 7<sup>th</sup> and 8<sup>th</sup> degrees of the Sytonon just scale above.

1/1 16/15 9/8 6/5 5/4 4/3 ?? 3/2 8/5 5/3 9/5 15/8 2/1

(Note that there is no readily available 5-limit tritone (25/18, 36/25)). Some of the simpler just tritones, invoking 7- and 11-limits are 10/7, 7/5, and 11/8 (all very different of course). Note the preference in this scale for *superparticular* ratios (where the

numerator = denominator +1) and the smallest possible numbers. These are often called epimores, and are a favorite harmonic idea of many composers.

I have deliberately used the two superparticular 5-limit major seconds here ( $9/8$  for the 2<sup>nd</sup> degree, and  $10/9$  below the  $2/1$ , or  $9/5$ , for the minor seventh), and of course  $10/9$  and  $16/9$  could be substituted just as well, depending on what kinds of fifths were required (in fact, the above scale has a “bad” supertonic 5<sup>th</sup>, remedied by either  $10/9$  for the 2<sup>nd</sup> degree, or  $27/16$  for the 6<sup>th</sup>).

### A 7-limit extended chromatic scale

Septimal scales are quite common throughout the world, but this one is simply “made up” to illustrate common septimal intervals.

1/1 21/20 8/7 7/6 5/4 4/3 10/7 3/2 5/3 12/7 7/4 40/21

This scale actually has two “major sixths” (the  $12/7$  is 33¢ wide of 12TET, the  $5/3$  16¢ narrow). Note the  $3/2$  between  $8/7$  (2) and  $12/7$  (6),  $7/6$  and  $7/4$ ,  $10/7$  and  $21/20$ , etc. Note also the similar problems with “limit collision”: the mediant P5<sup>th</sup> (3-7) is:

$$40/21 / 5/4 = 32/21$$

(29¢ sharp of the 12TET P5<sup>th</sup>, 27¢ sharp of  $3/2$ ).

### A fanciful 11-limit extended chromatic scale

This is similar to the septimal construction above in that the main “generators” of this scale are simple 1:5:3 (or 4:5:6, which is the same thing) triads constructed above the 11-limit, in what Partch calls “otonality,” as opposed to the “minor triad,” or descending construction (down a  $5/4$  then down a  $6/5$ ), which he calls a “utonality.” In other words, the scale below uses the 4:5:6 triad idea through the primes 7 and 11 (with the notable absence of the  $99/64$ , a wide P5<sup>th</sup>, which is just a  $9/8$  M2<sup>nd</sup> on the  $11/8$  “tritone”).

1/1 33/32 12/11 77/64 14/11 4/3 11/8 3/2 11/7 128/77 11/6 64/33

As a difficult exercise, write this scale as adjacency intervals!

### Well-temperament

Perhaps among the most interesting set of tuning systems are those which approximate some set of ideal intervals (like just ratios) in the context of a fixed set of pitches, keys, etc. These can be referred to as *well-temperaments*, though that term has a more specific historical meaning in western art music. But a general interpretation of the term could include meantone tunings, slendro and pelog, many other world music systems, and in fact, equal temperament itself.

In well-temperaments, ideal tunings (often, historically,  $5/4$  thirds and  $3/2$  fifths) are approximated by most or all of the respective scale degree intervals. Few or none of the intervals are exact. These tunings may be thought of as techniques for *fitting* a fixed number of pitches to a fixed set of tuning relationships.

**UP TO HERE...**

Historically, *meantone temperaments* preserve relationships exactly for a few intervals, and compromise others by a fixed amount (1/4 comma, 1/5 comma, 1/6 comma, and others). Well-temperaments usually have few (if any) exact ideal intervals — the “error” is distributed over the entire set. Consequently, well-temperaments are thought to be more in tune for a greater number of musical keys than meantones, which are usually “good” for some keys, and “less good” for others.

**Werckmeister III**

Rasch’s description and analysis of “Werckmeister III” (W3) is a good example of how well-temperaments work (historically). This tuning is often thought to be one of the major advances in historical tunings, and possibly the one intended by Bach in the WTC (Kirnberger was another historical tuning theorist who is a possibility here, and Young 2 is another late well-temperament which is thought to as good as W3).

The tuning is (rounded to the nearest cent):

C#	90¢
D	192¢
Eb	294¢
E	390¢
F	498¢
F#	588¢
G	696¢
G#	792¢
A	888¢
Bb	996¢
B	1092¢
C	1200¢

Oddly, the “best” key in W3, by at least one common measure, is F. According to Rasch, there are four tempered 5<sup>th</sup>s (each flat by about 6¢), three of which lie in the “center” of the tuning (G/C, D/G, A/D). All other 5<sup>th</sup>s are pure (3/2). For example, Bb/Eb = C#/F# = 702¢. None of the 3<sup>rd</sup>s are “pure” (5/4). The “central” ones are “rather good” but two wide (e.g E/C = 390¢), and the “peripheral ones” (from the key of C around the circle of fifths) Pythagorean (Bb/F# = 406¢, roughly equivalent to the 81/64 Pythagorean 3<sup>rd</sup>, about 21¢ wide of the 5/4 just).

“In such a tuning the central tonalities (with only a few or no sharps or flats) are rather good. The peripheral tonalities (with many sharps or flats) are not too bad, and tolerable, in any case. There is no clear distinction between ordinary and wolf intervals, like there is in meantone tuning. There are no wolves. All tones and intervals can be used enharmonically... With this tuning, Werckmeister has fulfilled his own demands to construct a tuning in which all tonalities could be performed without... the disturbing effects of wolf intervals”

Note that the evaluation is based on the 3<sup>rd</sup>s and 5<sup>th</sup>s (and by consequence, triads), which can be shown in the following way (the larger, bold intervals — the complete network of W3 intervals is also shown by this half-matrix):

	C	C#	D	E <sup>b</sup>	E	F	F#	G	A <sup>b</sup>	A	B <sup>b</sup>	B	C
C		90	192	294	<b>390</b>	498	588	<b>696</b>	792	888	996	1092	1200
C#			102	204	300	<b>408</b>	498	606	<b>702</b>	798	906	1002	1110
D				102	198	306	<b>396</b>	504	600	<b>696</b>	804	900	1008
E <sup>b</sup>					96	204	294	<b>402</b>	498	594	<b>702</b>	798	906
E						108	198	306	<b>402</b>	498	606	<b>702</b>	810
F							90	198	294	<b>390</b>	498	594	<b>702</b>
F#								108	204	300	<b>408</b>	504	612
G									96	192	300	<b>396</b>	504
G#										96	204	300	<b>408</b>
A											108	204	312
B <sup>b</sup>												96	204
B													108

(Note that the diagonals are intervals, the rows and columns “keys”).

Based on the idea of how well the tempered intervals match just 5<sup>th</sup>s and 3<sup>rd</sup>s, Rasch develops a number of measures for well-temperaments, including:

- 1) the “mean tempering of a major triad...the mean of the absolute values (in cents) of the temperings of the fifth, the major third, and the minor third in the triad”
- 2) the “mean tempering of a key... the weighted mean of the temperings of all triads” (triads are given successively lower weights starting from the tonic, and proceeding downward through the circle of fifths)
- 3) the “mean tempering of a tuning... the mean tempering of all consonant intervals, ...equal to the mean tempering of all triads, or of all keys”

Chalmers also uses the idea of “error” in his evaluation and creation of what he refers to as “linear temperaments” by the *Method of Least Squares*. Beginning with the just M3<sup>rd</sup> and P5<sup>th</sup>, he later extends his method to other ratios, finding new tunings which are “optimized” meantones (meantones with minimized errors). Chalmers also points out that the design of the error function is independent of the particular ratios desired. “There are innumerable ways in which the error functions can be combined. Various means, arithmetic, harmonic, geometric, to name the simplest, may be used.”

Both Chalmers and Rasch thus consider various temperaments as approximations, in some way, of ideal tunings. The error calculation (whatever that might specifically mean) is a first step towards designing a tuning which optimally fits some set of ratios, weights, and predefined error function.

These ideas are later explored by Polansky, Rockmore, Johnson, Repetto, and Pan “A Mathematical Model for Optimal Tuning Systems,” *Perspectives of New Music*, 47/1:69-110. Winter, 2009. By establishing a simple set of cultural and perceptual criteria for scale formation, this article explores the natural evolution of tuning systems via a mathematical model. One of this article’s underlying assertions is that well-temperament can serve as a model for all tunings, not just western historical ones.

### Harmonic Distance Functions

Many theorists have attempted to devise functions which measure, in some explainable way, relative consonance of intervals. These functions usually have three criteria:

- 1) smallness of the numbers
- 2) smallness of primes and prime powers
- 3) number of distinct factors

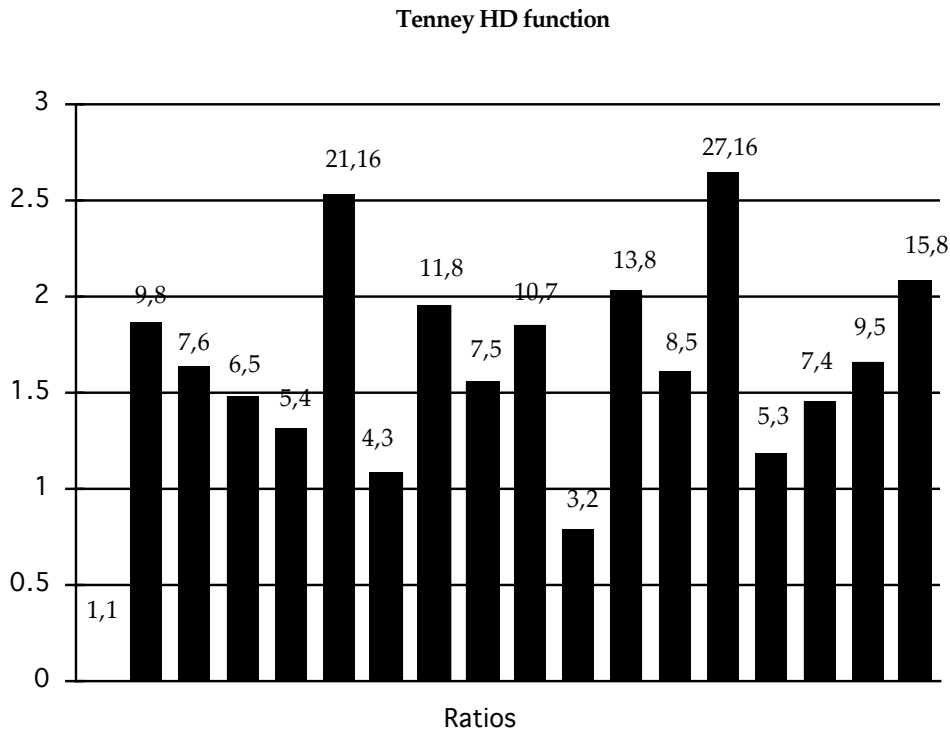
In other words, such a function typically finds  $3/2$  more consonant than  $5/4$ ,  $5/4$  than  $7/6$ , but has some trouble distinguishing between, for example  $10/8$  and  $9/8$  (both major seconds, one with a higher prime power of  $3^2$ , the other with a higher prime in the numerator).

Tenney's *HD* function, in one of its simplest forms, is perhaps the most direct. It is

$$HD(a/b) = \log(ab)$$

—where  $a/b$  is a relatively prime, octave reduced ratio. Chalmers describes this as a “special case of the Minkowski metric in a tonal space where the units along each of the axes are the logarithms of prime numbers.” Note that  $\log(ab) = \log(a) + \log(b)$ , showing the Minkowski-nature of this metric. Tenney's function is closely related to his concept of “harmonic space,” which is similar to the Johnston lattice, and other such graphic models of rational tunings.

The following chart shows a simple plot of Tenney's *HD* functions for a number of simple just ratios within the 2:1.



Other functions like this are Barlow's *harmonicity* and *digestibility* functions (described in his 1987 *Computer Music Journal* article), and the Euler GS (*gradus suavitatis*) function.

The Euler function is especially important, because of its simplicity, computability, and power. It yields similar results to the Tenney HD, and is defined as follows:

- the GS of a prime number is itself
- the GS of a composite number is the sum of the GSs of the prime factors, minus one less than the number of factors

In other words, the GS of 3 is 3. The GS of 9 is  $6 - (2-1) = 5$ . The GS of 45 ( $= 5*3*3$ ) is  $5 + 3 + 3 - 2 = 9$ .

GS values for ratios are treated similarly.  $3/2$  is  $3 + 2 - 1 = 4$ ;  $9/8 = 3*3/2*2*2 = 6 + 6 - 4 = 8$ ; GS of  $5/4$  is  $5 + 2 + 2 - 2 = 9 - 2 = 7$ , etc. The GS is only taken on relatively prime ratios. One interesting feature of the GS is that it emphasizes, in its construction, the criteria outlined above for harmonic distance functions in general (as does the Barlo function, but this is much more complicated to compute). One unusual aspect of the Tenney function, since it does not really factor "primeness" explicitly in its measure, is that it finds the  $M6^{\text{th}}$  to have a slightly smaller distance than the  $M3^{\text{rd}}$ , whereas the GS functions finds them equal (7, though the GS has a coarser resolution).

LP  
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